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MATHEMATICS

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1961

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Abstract

Full Text

MATHEMATICS

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ON THE CONVERGENCE OF DIFFERENCE SCHEMES FOR THE HEAT-CONDUCTION EQUATION WITH VARIABLE COEFFICIENTS

(Presented by Academician S. L. Sobolev on 31 III 1961)

1. For the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^m \frac{\partial}{\partial x_i} \left[\chi(x_1, x_2, \dots, x_m, t) \frac{\partial u}{\partial x_i} \right], \quad \chi(x_1, x_2, \dots, x_m, t) \geq \chi_0 > 0; \quad (1)$$

we pose the mixed Cauchy problem

$$u(x_1, x_2, \dots, x_m, 0) = \varphi(x_1, x_2, \dots, x_m); \quad (2)$$

$$\begin{aligned} u(x_1, x_2, \dots, x_{s-1}, 0, x_{s+1}, \dots, x_m, t) &= f_s(x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_m, t), \\ u(x_1, x_2, \dots, x_{s-1}, 1, x_{s+1}, \dots, x_m, t) &= g_s(x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_m, t). \end{aligned} \quad (3)$$

We shall require sufficient smoothness of the functions $\chi(x, t)$, $\varphi(x)$, $f_s(x, t)$, $g_s(x, t)$.

For the solution of problem (1)–(3) we construct the following difference scheme:

$$A_2(\alpha r^{n+1})u^{n+1} = A_1[(1 - \alpha)r^n]u^n, \quad 0 \leq \alpha \leq 1; \quad (4)$$

$$u_{i_1 \dots i_m}^0 = \varphi_{i_1 \dots i_m} = \varphi(i_1 h_1, \dots, i_m h_m); \quad (5)$$

$$u_{i_1 \dots i_{s-1}, 0, i_{s+1} \dots i_m}^n = f_{s i_1 \dots i_{s-1}, i_{s+1} \dots i_m}^n, \quad (6)$$

$$u_{i_1 \dots i_{s-1}, N_s+1, i_{s+1} \dots i_m}^n = g_{s i_1 \dots i_{s-1}, i_{s+1} \dots i_m}^n,$$

where it is put that

$$u^n = \{u_{i_1 \dots i_m}^n\} = u(i_1 h_1, \dots, i_m h_m, n\tau);$$

$$i_s = 0, 1, \dots, (N_s + 1); \quad (N_s + 1)h_s = 1;$$

$$r^n = \{r_{s i_1 \dots i_m}^n\}; \quad r_{s i_1 \dots i_m}^n = \frac{\chi_{i_1 \dots i_s + 1/2 \dots i_m}^n \tau}{h_s^2};$$

$$A_2(\alpha r) = E - \alpha B(r); \quad A_1[(1 - \alpha)r] = E + (1 - \alpha)B(r);$$

$$B(r) = \sum_{s=1}^m B_s; \quad B_s = \{B_{s i}^j\}; \quad (7)$$

$$\begin{aligned} B_{s i}^j &= r_{s i_1 \dots i_s - 1 \dots i_m} \delta_{i_1 \dots i_s - 1 \dots i_m}^{j_1 \dots j_s \dots j_m} \\ &\quad - (r_{s i_1 \dots i_s - 1 \dots i_m} + r_{s i_1 \dots i_s \dots i_m}) \delta_{i_1 \dots i_s \dots i_m}^{j_1 \dots j_s \dots j_m} \\ &\quad + r_{s i_1 \dots i_s \dots i_m} \delta_{i_1 \dots i_s + 1 \dots i_m}^{j_1 \dots j_s \dots j_m}, \end{aligned}$$

$$\delta_{i_1 \dots i_m}^{j_1 \dots j_m} = \delta_{i_1}^{j_1} \dots \delta_{i_m}^{j_m}, \quad \delta_i^j \text{ is the Kronecker symbol.}$$

By the equivalence theorem ⁽¹⁾, for convergence of the difference scheme (4)–(6) it is necessary and sufficient that the step operator of equation (4)

$$C_n = A_2^{-1}(\alpha r^{n+1}) A_1[(1 - \alpha)r^n] \quad (8)$$

have a norm satisfying the condition

$$\|C_n\| = 1 + d_n \tau, \quad |d_n| \leq K; \quad (9)$$

the constant K does not depend on n, τ, h . We shall show that in our case this is so.

§ 2. We first formulate a number of lemmas.

Lemma 1. Let A, B be two symmetric matrices; let $C(\varepsilon)$ be the matrix of the pencil

$$C(\varepsilon) = A + \varepsilon B; \quad (10)$$

$x_i(\varepsilon)$ the i -th normalized eigenvector of $C(\varepsilon)$; $\lambda_i(\varepsilon)$ the corresponding eigenvalue.

Then $x_i(\varepsilon)$, $\lambda_i(\varepsilon)$ satisfy the following system of equations

$$\frac{d\lambda_i}{d\varepsilon} = (Bx_i, x_i); \quad (11)$$

$$\frac{dx_i}{d\varepsilon} = \sum_{k=1, k \neq i}^n \frac{(Bx_i, x_k)}{\lambda_i - \lambda_k} x_k. \quad (12)$$

Lemma 2. If

$$r_{si_1 \dots i_m} \geq r_0 > 0, \quad 0 \leq \alpha \leq 1, \quad (13)$$

then the matrix $B(r)$ has nonnegative characteristic roots, the matrix $A_2(\alpha r)$ has positive roots not less than 1, and the matrix $A_1[(1 - \alpha)r]$ has roots not exceeding 1.

Lemma 2 follows from Lemma 1.

Lemma 3. Let

$$r_s = \min_{i_1 \dots i_m} \{r_{si_1 \dots i_m}\}, \quad R_s = \max_{i_1 \dots i_m} \{r_{si_1 \dots i_m}\}; \quad 0 \leq \alpha \leq 1,$$

$$b_s = B(r_s); \quad B_s = B(R_s), \quad b = \sum_{s=1}^m b_s, \quad B = \sum_{s=1}^m B_s; \quad (14)$$

$$a_2 = E - \alpha b; \quad a_1 = E + (1 - \alpha)b; \quad c = a_2^{-1}a_1;$$

$$A_2 = E - \alpha B, \quad A_1 = E + (1 - \alpha)B, \quad C = A_2^{-1}A_1.$$

Then for the matrix

$$C(r) = A_2^{-1}(\alpha r)A_1[(1 - \alpha)r] \quad (15)$$

the relation

$$\|c\| \leq \|C(r)\| \leq \|C\| \quad (16)$$

is valid.

Lemma 4. Put

$$\tilde{C}_n = A_2^{-1}(\alpha r^n)A_1[(1 - \alpha)r^n]. \quad (17)$$

Then for C_n (8) one has

$$\|C_n\| = \|\tilde{C}_n\|(1 + l_n\tau), \quad |l_n| \leq K, \quad (18)$$

where K does not depend on n, τ, h .

On the basis of Lemmas 1-4 one may formulate the following theorem.

Theorem 1. The step operator C_n of scheme (4) is stable if the operator C is stable, and unstable if the operator c is unstable.

Since c, C are operators with constant coefficients, the Neumann criterion is applicable for determining their stability. Therefore the following sufficient convergence criterion can be formulated:

Theorem 2. If $1/2 \leq \alpha \leq 1$, or $0 \leq \alpha < 1/2$ and

$$\sum_{s=1}^m R_s \leq \frac{1}{2 - 4\alpha}, \quad (19)$$

then the solution of (4)–(6) converges to the solution of (1)–(3) in the mean.

Let us note that Lemma 3 and Theorems 1 and 2 are also valid for nonlinear equations. However, in this case the R_s depend on the initial data and the mesh and cannot be estimated effectively.

The indicated method is also applicable to hyperbolic equations.

Received
2 I 1961

References

1. R. D. Richtmyer, *Difference Methods for Initial-Value Problems*, 1960, p. 57.

Note: Figure translations are in progress. See original paper for figures.

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