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Abstract

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MATHEMATICS

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CONSTRUCTION OF TRANSITION SOLUTIONS OF NONLINEAR EQUATIONS

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1. The study of the structure of shock waves in ordinary ($^{1-3}$) and magnetic ($^{4-8}$) hydrodynamics leads to the following problem: to find that solution of a system of ordinary differential equations which, as $x \to \pm\infty$, tends to prescribed limits. The same problem arose in the theory of hyperbolic systems of quasilinear equations in an attempt to determine a class of functions in which the Cauchy problem has a unique solution (^{9}). In the present note an analogous question is considered for one differential equation of arbitrary order.

A solution $y(x)$ of the nonlinear equation

$$y^{(n)}(x) = F(x, y, y', \dots, y^{(n-1)})$$

is called a **transition** solution if it tends to prescribed limits q_1 and q_2 as $x \to -\infty$ and $x \to +\infty$, respectively, and

$$\lim_{x\to\pm\infty} y^{(k)}(x) = 0,$$

$k = 1, 2, \dots, n$. Without restricting the generality of the consideration, we shall assume that $q_1 < 0,\ q_2 > 0$, and $y(0) = 0$.

2. Consider the quasilinear equation

$$P_0\left(\frac{d}{dx}\right) y + f(x, y) = 0, \qquad \text{(A)}$$

where

$$P_0(\nu) = a_n\nu^n + a_{n-1}\nu^{n-1} + \dots + a_1\nu \quad (a_1 < 0,\ a_n \neq 0).$$

Everywhere in what follows we shall assume that **all roots of the polynomial** $P_0(\nu)$ **are real and simple**. Denote by $a_-(> 0)$ and $a_+(< 0)$ any two numbers such that the polynomials

$$P_\pm(\nu) = P_0(\nu) + a_\pm$$

also have only real and simple roots:

$$P_-(\nu) = a_n \prod_{j=1}^{l} (\nu - \lambda_j^-) \prod_{j=1}^{n-l} (\nu - \mu_j^-), \qquad P_+(\nu) = a_n \prod_{j=1}^{l+1} (\nu - \lambda_j^+) \prod_{j=1}^{n-l-1} (\nu - \mu_j^+);$$

all the numbers $\lambda_j^{\pm}$ are negative, and the $\mu_j^{\pm}$ are positive.

With respect to the function $f(x, y)$ we shall assume:

1) There exist two numbers $p_1 (\leq q_1)$ and $p_2 (\geq q_2)$ such that

$$f(x, p_1) \leq 0 \quad (-\infty < x \leq 0), \qquad f(x, p_2) \leq 0 \quad (0 \leq x < \infty),$$

and

$$p_1 a_- = p_2 a_+.$$

2) In the closed domains

$$D_- \, (-\infty < x \leq 0, \; p_1 \leq y \leq 0)$$

and

$$D_+ \, (0 \leq x < \infty, \; 0 \leq y \leq p_2)$$

the function $f(x, y)$ is defined and is uniformly continuous in y with respect to x and y.

3)

$$f(x, 0) > 0, \qquad -\infty < x < \infty.$$

4)

$$\frac{f(x, y_2) - f(x, y_1)}{y_2 - y_1} \begin{cases} < a_-, & (x, y_1), (x, y_2) \in D_-, \\ > a_+, & (x, y_1), (x, y_2) \in D_+. \end{cases}$$

5) There exist limits, uniform with respect to y,

$$f(-\infty, y) = \lim_{x \to -\infty} f(x, y) \qquad (p_1 \leq y \leq 0);$$

$$f(\infty, y) = \lim_{x \to +\infty} f(x, y) \qquad (0 \leq y \leq p_2).$$

6) The function $f(-\infty, y)$ $(f(\infty, y))$ has on the interval $(p_1, 0)$ $((0, p_2))$ a unique zero $y = q_1$ $(y = q_2)$.

We denote by $G(q_1, q_2)$ the set of all functions $f(x, y)$ possessing properties 1)–6).

We denote by S the set of all functions $\omega(x)$, continuous on the closed intervals $[-\infty, 0]$ and $[0, \infty]$, such that the points $(x, y = \omega(x))$ lie in the domain $D = D_- + D_+$ for all x $(-\infty < x < \infty)$. We shall be interested only in those transition solutions $y(x)$ which belong to S.

Introduce the following notation:

$$a(\xi) = \begin{cases} a_-, & \xi \leqslant 0, \\ a_+, & \xi > 0; \end{cases} \qquad \varphi(x, y) = y a(y) - f(x, y).$$

If $y(x) \in S$, then, evidently, $a(y(x)) \equiv a(x)$. Therefore equation (A) may be rewritten as

$$P_0\left(\frac{d}{dx}\right) y + a(x)y = \varphi(x, y), \qquad y \in S.$$

It follows that

$$y(x) = \int_{-\infty}^{\infty} K(x, s)\varphi(s, y(s))\, ds, \tag{1}$$

where $K(x, s)$ is the Green’ s function of the linear operator $P_\theta(d/dx) + a(x)$, singled out by the conditions $K(\pm\infty, s) = K(0, s) = 0$ $(-\infty < s < \infty)$.

3. Rewrite the equation defining the Green’ s function

$$P_0\left(\frac{d}{dx}\right) K(x, s) + a(x)K(x, s) = \delta(x - s),$$

in the form

$$P_-\left(\frac{d}{dx}\right) K(x, s) = \delta(x - s) + (a_- + a_+)\sigma(x)K(x, s), \qquad \sigma(x) = \frac{1 + \operatorname{sgn} x}{2}.$$

Denote by $Y_-(x - s)$ the Green’ s function of the operator $P_-(d/dx)$, singled out by the conditions $Y_-(\pm\infty) = 0$. The last equation is, evidently, equivalent to the integral equation on a half-axis with difference kernel

$$K(x, s) = Y_-(x - s) + (a_- - a_+) \int_0^{\infty} Y_-(x - t)K(t, s)\, dt.$$

Solving it, after simple transformations we obtain

$$K(x, s) = \frac{a_- - a_+}{a_n^2} \int_0^{x} Y(\lambda^+, \mu^-, x - t)\, Y(\lambda^-, \mu^+, t - s)\, dt, \tag{2}$$

where $Y(\lambda^+, \mu^-, x - t)$ and $Y(\lambda^-, \mu^+, t - s)$ are the Green’ s functions of the operators $\prod(d/dx - \lambda_i^+) \prod(d/dx - \mu_j^-)$ and $\prod(d/dx - \lambda_i^-) \prod(d/dx - \mu_j^+)$, respectively. The functions $Y(\lambda^+, \mu^-, t)$ and $Y(\lambda^-, \mu^+, t)$ are of constant sign and have opposite signs, since the number of positive roots μ^- differs by one from the number of positive roots μ^+. Therefore

$$xK(x, s) \leqslant 0, \qquad -\infty < x, s < \infty. \tag{3}$$

Relying on formula (2), it is easy to show that

$$a(x) \int_{-\infty}^{\infty} K(x,s)\, ds < 1, \qquad -\infty < x < \infty. \tag{4}$$

4. Consider the operator

$$H\omega(x) = \int_{-\infty}^{\infty} K(x,s)\, \varphi(s, \omega(s))\, ds,$$

defined on functions $\omega \in S$. Equation (1) can be rewritten in the form $y = Hy$.

Owing to properties (3) and (4) of the kernel $K(x,s)$, the set of functions S is mapped by the operator H into itself. We shall say that a function $\omega_1 \in S$ is *steeper* than a function $\omega_2 \in S$ ($\omega_1 \succcurlyeq \omega_2$, $\omega_2 \preccurlyeq \omega_1$), if their difference $\omega_1 - \omega_2$ belongs to S. It is easy to show, relying on the properties of the function $f(x,y)$, that if $\omega_1 \succcurlyeq \omega_2$, then $H\omega_1 \succcurlyeq H\omega_2$, i.e. the operator H is monotone.

Among the functions $\omega \in S$ there is a "least steep" function $\Omega_0(x) \equiv 0$ and a "most steep" function $\omega_0(x)$: $\omega_0(x) = p_1$ $(x < 0)$, $\omega_0(x) = p_2$ $(x > 0)$. It is clear that $H\Omega_0 \succcurlyeq \Omega_0$ and $H\omega_0 \preccurlyeq \omega_0$. Therefore, putting $\omega_n = H^n\omega_0$ and $\Omega_n = H^n\Omega_0$, we shall have

$$\Omega_0 \preccurlyeq \Omega_1 \preccurlyeq \Omega_2 \preccurlyeq \ldots \preccurlyeq \Omega_n \preccurlyeq \ldots \preccurlyeq \omega_n \preccurlyeq \omega_{n-1} \preccurlyeq \ldots \preccurlyeq \omega_1 \preccurlyeq \omega_0. \tag{5}$$

It is now not difficult to show that the sequence Ω_n (ω_n) converges uniformly on the entire axis to some function Ω (ω), and the function $y(x) = \Omega(x)$ ($y = \omega(x)$) belongs to S and satisfies the equation $y = Hy$.

Thus, the following assertions can be obtained:

Theorem 1. *If $f(x,y) \in G(q_1, q_2)$, then equation (A) has in S at least one transition solution. Among the transition solutions of equation (A) (if there are several) there exists a most steep solution*

$$\omega(x) = \lim_{n\to\infty} H^n \omega_0(x)$$

and a least steep solution

$$\Omega(x) = \lim_{n\to\infty} H^n \Omega_0(x).$$

We emphasize that the arguments proving this theorem are very close to the arguments used by P. S. Uryson in his work (10), devoted to the investigation of nonlinear integral equations of a special type.

Theorem 2. *If the function $f(x,y) \in G(q_1, q_2)$ in equation (A) is replaced by a larger function $f_1(x,y) \succcurlyeq f(x,y)$ $((x,y) \in D,\ f_1 \in G(q_1^0, q_2^0),\ q_1^0 \preccurlyeq q_1,\ q_2^0 \succcurlyeq q_2)$, then the most and least steep solutions $\omega(x)$ and $\Omega(x)$ become steeper.*

Relying on Schauder' s principle (see, for example, (11)) and Theorems 1 and 2, it is easy to prove the following theorem:

Theorem 3. *If the function f depends only on y: $f = f(y)$, satisfies the conditions*

$$f(y) > 0, \qquad q_1 < y < q_2, \qquad f(y) < \begin{cases} a_-(y - q_1), & q_1 \le y \le 0, \\ a_+(y - q_2), & 0 \le y \le q_2, \end{cases}$$

and is continuous on the interval $[q_1, q_2]$, then equation (A) has a monotone transition solution.

5. Let $f(x, y) = f(y) + \psi(x) \in G(q_1, q_2)$, and suppose that within the intervals $(p_1, 0)$ and $(0, p_2)$ the function $f(y)$ is monotone, while outside some interval $\alpha < y < \beta$ ($q_1 < \alpha < 0 < \beta < q_2$) it satisfies the condition

$$\left| \frac{f(y_2) - f(y_1)}{y_2 - y_1} \right| > d, \qquad \alpha < y_1 < y_2 < \beta, \tag{6}$$

where d is some positive number.

In this case equation (A) has a unique solution in S. Denote by $\Psi(f)$ the totality of all functions ψ satisfying the condition $f(y) + \psi(x) \in G(q_1^0, q_2^0)$ ($q_1^0 \curlyeqsucc p_1$, $q_2^0 \curlyeqprec p_2$). To each function $\psi \in \Psi(f)$ we assign the solution $y(x)$ of equation (A): $y(x) = T\psi(x)$.

Theorem 4. The operator T is continuous if the functions $y(x) \in S$ and $\psi(x) \in \Psi(f)$ are regarded as elements of the space $C(-\infty, \infty)$.

The operator T is monotone, i.e. $T\psi_1 \ge T\psi_2$, if $\psi_1 \ge \psi_2$, ($\psi_1, \psi_2 \in \Psi(f)$).

Relying on Schauder' s principle and these properties of the operator T, one can prove the following theorem.

Theorem 5. The equation $P_0(d/dx)y + f(y) + V(y) = 0$ has a transition solution in S, if:

1) The function $f(y)$ is defined, continuous, and monotone on the closed intervals $[p_1, 0]$ and $[0, p_2]$ and satisfies condition (6).

2)

$$\frac{f(y_2) - f(y_1)}{y_2 - y_1} \begin{cases} < a_-, & y_1, y_2 < 0, \\ > a_+, & y_1, y_2 > 0, \end{cases} \qquad f(q_1) = f(q_2) = 0.$$

3) $V(y) \ge 0$ ($q_1 \le y \le q_2$); $V(y) = 0$ ($y \notin (q_1, q_2)$).

4) The function $V(y)$ is continuous in the intervals $p_1 < y \le 0$ and $0 \le y < p_2$.

5)

$$\max_{p_1 \le y \le 0} V(y) \le -f(p_1), \qquad \max_{0 \le y \le p_2} V(y) \le -f(p_2).$$

Theorem 6. The equation

$$P_0(d/dx)y + f(y) + \varepsilon\Phi(y, y', \dots, y^{(n-1)}) = 0$$

has a transition solution if the function $f(y)$ satisfies the same conditions as in Theorem 5, ε is a sufficiently small positive number, and the function Φ has a bounded gradient in some domain of the variables $y, y', \dots, y^{(n-1)}$, depending on the form of the function $f(y)$.

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