MATHEMATICS

S. I. GELFAND and D. A. KAZHDAN

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. I. GELFAND and D. A. KAZHDAN

ON AN INTEGRAL EQUATION CONNECTED WITH THE MOTION OF A PULSE AROUND A CIRCLE

(Presented by Academician I. G. Petrovskii on 19 VI 1961)

This note is devoted to the following problem. Suppose that we have a circle along which a point pulse propagates. The speed of propagation of the pulse depends on the state of the given point of the circle, which is characterized by the time τ elapsed since the last passage of the pulse through this point of the circle. Then

$$v = c(\tau), \tag{1}$$

where $c(\tau)$ is a given function, continuous and monotonically increasing. Our aim is to prove that if we prescribe τ on the circle arbitrarily and then send one or several pulses around the circle, then their speed will tend to a certain constant independent of the initial state of the circle. If the number of pulses is greater than one, then the speed to which they tend will be the same for all pulses, and they will settle at equal distances from one another. This problem was posed in the paper (^{1}).

We shall prove our assertion for the case of one pulse. First let us formulate the problem somewhat differently. Unroll the circle onto a straight line, i.e., we shall represent each revolution of the pulse by a segment of a straight line. Suppose the pulse is at the point x of the line. Then its speed is $v = c[\tau(x)]$, where $\tau(x)$ is the time elapsed since the last excitation of the same point of the circle, i.e., the point $x-1$ of the line (we assume, for simplicity, that the circle has unit length). More precisely, $\tau(x) = t(x) - t(x-1)$, where $t(x)$ is the time elapsed from the beginning of the motion of the pulse to its arrival at the point x. We must prove that $\tau(x)$ tends to a limit as $x \to \infty$. To solve the problem we shall have to assume that the function $c(\tau)$ is strictly monotone and differentiable everywhere.

We shall show that for $x \geqslant 1$ the function $\tau(x)$ satisfies the integral equation:

$$\int_{x-1}^{x} \frac{dy}{c[\tau(y)]} = \tau(x). \tag{2}$$

Indeed, by the definition of velocity we have $v = c[\tau(x)]$, i.e. $t'(x) = c^{-1}[\tau(x)]$. Integrating, we obtain

$$\int_{x-1}^{x} \frac{dy}{c[\tau(y)]} = t(x) - t(x-1) = \tau(x).$$

It is clear that if the function $\tau(x)$ is given for $0 \leqslant x < 1$, then the solution of equation (2) exists and is unique for all $x \geqslant 1$. The aim of our note is to prove the following theorem:

Theorem. *Let $\tau(x)$ be a solution of the integral equation (2). Then, as $x \to \infty$, the function $\tau(x)$ tends to a finite limit.*

Proof. First we derive from (2) some formulas needed below. Differentiating equality (2), we obtain

$$\tau'(x) = c^{-1}[\tau(x)] - c^{-1}[\tau(x-1)] \qquad (x \geqslant 1). \tag{3}$$

Applying formula (2) to p successive revolutions and adding the equalities obtained, we have

$$\int_{x-p}^{x} \frac{dy}{c[\tau(y)]} = \sum_{k=0}^{p-1} \tau(x-k) \qquad (x \geqslant p-1). \tag{4}$$

Differentiating equality (3), we easily obtain that $|\tau''(x)|$ is bounded.

We now have all the formulas needed for the solution of the problem. Divide the half-line $x \geqslant 0$ into intervals of length 1 with integral endpoints, and call the interval $[n-1, n]$, where n is an integer, the n-th interval. Denote by M_n the greatest, and by m_n the least value of the function $\tau(x)$ on the n-th interval, and prove that the numbers M_n do not increase, while the m_n do not decrease, as n grows.

Let us prove this for the numbers M_n. Denote by x'_n the point at which the greatest value of $\tau(x)$ on the n-th interval is attained. We must show that $M_{n-1} \geqslant M_n$. It is necessary to distinguish two cases.

First let $\tau'(x'_n) \geqslant 0$. Then, by formula (3), we have $c^{-1}[\tau(x'_n)] - c^{-1}[\tau(x'_n - 1)] \geqslant 0$, and, in view of the fact that the function $c^{-1}(\tau)$ decreases, we obtain $\tau(x'_n - 1) \geqslant \tau(x'_n) = M_n$, and hence $M_{n-1} \geqslant \tau(x'_n - 1) \geqslant M_n$.

Now let $\tau'(x'_n) < 0$. Then the point x'_n is the left endpoint of the n-th interval, and therefore the right endpoint of the $(n-1)$-st interval; that is, in this case as well, $M_{n-1} \geqslant \tau(x'_n) = M_n$.

Thus we have shown that the sequence $\{M_n\}$ is nonincreasing. In the same way one can show that the sequence $\{m_n\}$ is nondecreasing. But it is clear that both these sequences are bounded; therefore they have limits M and m.

If we now prove that $\lim M_n = \lim m_n$ $(n \to \infty)$, i.e. that $M = m$, then the theorem will be proved. For the proof we shall need the following:

Lemma. Denote by x_n' the point at which the greatest value, and by x_n'' the point at which the least value, of $\tau(x)$ is attained on the n-th interval. Then, for any fixed integer $p \geqslant 0$, the differences $\tau(x_n' - p) - M_n$ and $\tau(x_n'' - p) - m_n$ tend to zero as $n \to \infty$.

We prove this assertion for maxima (for minima the proof is analogous). We shall argue by induction. For $p = 0$ the assertion is obvious. Suppose that $\tau(x_n'-p)-M_n \to 0$ as $n \to \infty$. We shall prove then that also $\tau(x_n'-p-1)-M_n \to 0$ as $n \to \infty$. To this end we first show that $\tau'(x_n' - p) \to 0$ as $n \to \infty$. Put $|\tau'(x_n' - p)| = \varepsilon_n$. Now take, for some n, an interval of length $\varepsilon_n/2\alpha$, where $\alpha = \max|\tau''(x)|$, in such a way that one of its endpoints is the point $x_n' - p$, and the interval itself is directed toward the increase of $\tau(x)$. Let the other endpoint of the interval be the point x_0. Then, for any point ξ belonging to the interval $[x_n' - p, x_0]$, we shall have $\tau'(\xi) > \varepsilon_n/2$. Hence it is clear that $\tau(x)$ changes on the interval $[x_n' - p, x_0]$ by more than

$$\frac{\varepsilon_n}{2\alpha}\,\frac{\varepsilon_n}{2} = \frac{\varepsilon_n^2}{4\alpha},$$

i.e. $\tau(x_0) - \tau(x_n' - p) > \varepsilon_n^2/4\alpha$. But $\tau(x_0) \leqslant M_{n-p-1}$, and therefore

$$M_{n-p-1} - \tau(x_n' - p) = M_{n-p-1} - M_n + M_n - \tau(x_n' - p) > \frac{\varepsilon_n^2}{4\alpha},$$

and hence $\varepsilon_n \to 0$ as $n \to \infty$, since $M_{n-p-1} - M_n \to 0$ and $M_n - \tau(x_n' - p) \to 0$ as $n \to \infty$.

It is now easy to show that $\tau(x_n' - p - 1) - M_n \to 0$. Indeed,

since $\tau'(x_n - p) \to 0$, we have, applying (3), $c^{-1}[\tau(x_n' - p)] - c^{-1}[\tau(x_n' - p - 1)] \to 0$, and hence, by the strict monotonicity of $c^{-1}(\tau)$, $\tau(x_n' - p - 1) - \tau(x_n' - p) \to 0$ as $n \to \infty$. Consequently, $\tau(x_n' - p - 1) - M_n = \tau(x_n' - p - 1) - \tau(x_n' - p) + \tau(x_n' - p) - M_n \to 0$ as $n \to \infty$. Thus we have proved that $\tau(x_n' - p) - M_n \to 0$ as $n \to \infty$. It is proved similarly that also $\tau(x_n'' - p) - m_n \to 0$ as $n \to \infty$. Thus the lemma is proved.

We now turn to the proof of the theorem, i.e., we shall prove that $M = m$. Suppose the contrary, namely suppose that $M > m$. Then one can find a natural number p and an $\varepsilon > 0$ such that $p(M - m - 2\varepsilon) > 2c^{-1}(m_1)$, where m_1 is the minimum of $\tau(x)$ on the first interval, and hence also the absolute minimum of $\tau(x)$. By virtue of the lemma just proved, one can find an n such that for all natural $k < p$ the inequalities $|\tau(x_n' - k) - M_n| < \varepsilon$ and $|\tau(x_n'' - k) - m_n| < \varepsilon$ will hold. Now write equality (4) for the points x_n' and x_n''. We obtain

$$\int_{x_n'-p}^{x_n'} \frac{dy}{c[\tau(y)]} = \sum_{k=0}^{p-1} \tau(x_n' - k); \qquad \int_{x_n''-p}^{x_n''} \frac{dy}{c[\tau(y)]} = \sum_{k=0}^{p-1} \tau(x_n'' - k)$$

and, subtracting the second equality from the first,

$$\begin{aligned}\int_{x_n'-p}^{x_n'} \frac{dy}{c[\tau(y)]} - \int_{x_n''-p}^{x_n''} \frac{dy}{c[\tau(y)]} &= \int_{x_n'-p}^{x_n''-p} \frac{dy}{c[\tau(y)]} - \int_{x'n}^{x_n''} \frac{dy}{c[\tau(y)]} \\ &= \sum_{k=0}^{p-1} \tau(x_n'-k) - \tau(x_n''-k).\end{aligned} \tag{5}$$

We shall show that this equality cannot hold, since its left-hand side is smaller than its right-hand side. To this end let us estimate them. We have

$$\int_{x_n'-p}^{x_n''-p} \frac{dy}{c[\tau(y)]} - \int_{x'n}^{x_n''} \frac{dy}{c[\tau(y)]} \leq 2c^{-1}(m_1),$$

since the length of each of the intervals of integration is $|x_n' - x_n''| < 1$, and the maximum of the integrand is equal to $c^{-1}(m_1)$. On the other hand, estimating the right-hand side of equality (5), we obtain

$$\sum_{k=0}^{p-1} \tau(x_n'-k) - \tau(x_n''-k) \geq \sum_{k=0}^{p-1} (M_n - \varepsilon) - (m_n + \varepsilon) \geq p[(M-\varepsilon) - (m+\varepsilon)] =$$

$$= p(M - m - 2\varepsilon) > 2c^{-1}(m_1) \geq \int_{x_n'-p}^{x_n''-p} \frac{dy}{c[\tau(y)]} - \int_{x'n}^{x_n''} \frac{dy}{c[\tau(y)]},$$

which contradicts (5). Thus our assumption that $M > m$ is false, i.e. $M = m$, and the function $\tau(x)$, and with it also $v = c[\tau(x)]$, tends to a limit as $x \to \infty$. The theorem is proved.

Finally, let us show that the limiting velocity of the pulse is unique and does not depend on the initial state of the ring, i.e. on the assignment

function $\tau(x)$ for $0 \leq x < 1$. Indeed, suppose that the impulse moves along the ring with constant velocity v_0. Then it completes one revolution in the time $\tau_0 = l v_0^{-1}$ (l is the length of the ring). But, by formula (1), $v_0 = c(\tau_0) = c(l v_0^{-1})$. However, this equation with respect to v_0 has a unique solution, since the function $c(l y^{-1})$ decreases together with $l y^{-1}$, while the function y increases. It is also clear that this solution does not depend on the initial specification of $\tau(x)$ on the ring.

The case in which the number of impulses is greater than one is treated similarly. In this case functions $\tau_i(x)$, analogous to the function $\tau(x)$, are introduced by the following formulas:

$$\tau_1(x) = t_1(x) - t_k(x-1); \quad \tau_2(x) = t_2(x) - t_1(x), \quad \tau_k(x) = t_k(x) - t_{k-1}(x),$$

where k is the number of impulses, and $t_i(x)$ is the time elapsed from the beginning of the motion of the impulses until the arrival of the i-th impulse at the point x. For $x \geq 1$, the functions $\tau_i(x)$ satisfy the system of differential equations

$$\tau_1'(x) = c^{-1}[\tau_1(x)] - c^{-1}[\tau_k(x-1)]$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\tau_k'(x) = c^{-1}[\tau_k(x)] - c^{-1}[\tau_{k-1}(x)].$$

These equations can also be written in integral form

$$\tau_1(x) + \tau_2(x) + \ldots + \tau_k(x) = \int_{x-1}^{x} \frac{dy}{c[\tau_k(y)]},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{6}$$

$$\tau_1(x) + \tau_2(x-1) + \ldots + \tau_k(x-1) = \int_{x-1}^{x} \frac{dy}{c[\tau_1(y)]}.$$

Just as was proved in the case of a single impulse, it can be proved that the functions $\tau_i(x)$ that are solutions of equations (6) tend to a limit as $x \to \infty$. It is also clear that these limits must be the same for all the functions, i.e., that the impulses become established at equal distances from one another.

Received
7 VI 1961

REFERENCES CITED

[1] I. M. Gel' fand, M. L. Tsetlin, DAN, **131**, No. 6 (1960).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.