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Abstract

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SOME EXAMPLES OF GROUPS OF AUTOMORPHISMS

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1. We shall use the definitions from [1]. Let G be a group, Σ some set of its automorphisms, and $\{\Sigma\}$ the group generated by this set. The set Σ is called **nilpotent** (relative to G) if, for every $g \in G$, there exists an $n = n(g)$ such that the equality

$$[g; \sigma_1, \sigma_2, \dots, \sigma_n] = 1,$$

holds, where $\sigma_i \in \Sigma$. In particular, the set Σ may consist of a single element, and then this element is called a **nil-automorphism**. The automorphism group Φ of the group G is called **externally locally nilpotent** if every finite set of its elements is nilpotent.

The set Σ is called **periodic** (relative to G) if G has a local system of Σ-admissible subgroups with a finite number of generators, in each of which $\{\Sigma\}$ induces a finite group of automorphisms. If all these induced groups of automorphisms are Π-groups, where Π is some set of prime numbers, then Σ will be called a **Π-set**. In particular, when Σ consists of a single element, we obtain a **periodic automorphism** (a Π-automorphism). The automorphism group Φ of the group G is called **generalized periodic** (a generalized Π-group) if all its elements are periodic automorphisms (Π-automorphisms). Finally, the group Φ is called **externally locally periodic** (an externally locally Π-group) if every finite set of its elements is periodic (a Π-set).

The main result of the note consists of the following propositions.

A. There exists a group of nil-automorphisms of an abelian group which is not externally locally nilpotent.

B. There exists a generalized periodic (even periodic in the ordinary sense) automorphism group of an abelian group which is not externally locally periodic.

C. There exists an automorphism group of an abelian group such that it is generated by its externally locally nilpotent normal divisor and a nil-automorphism, but itself is not an externally locally nilpotent group (not even a group of nil-automorphisms).

D. There exists an automorphism group of an abelian group such that it is generated by its externally locally periodic normal divisor and a periodic automorphism, but itself is not an externally locally periodic group (not even generalized periodic).

2. Lemma 1. Let G be a locally finite p-group. Then every finite nilpotent set of automorphisms of the group G is a p-set, and conversely.

Using this lemma, in proving propositions A—D we can get by with only two examples.

3. An example proving propositions A and B. Let A and Φ be two arbitrary groups, and let $U = A \wr \Phi$ be the wreath product of these groups ···

groups (see, for example, ([1])). The group Φ may be regarded as a group of automorphisms of the group $G = \prod_{\alpha\in\Phi} A_\alpha$.

Take as the group Φ an infinite group with a finite number of generators, satisfying the identity relation $x^p = 1$, where p is a prime number (the existence of such a group follows from the results of P. S. Novikov ([2])), and as the group A take the cyclic group of order p. It is not hard to see that Φ is a group of nil-automorphisms of the group G. Indeed, if $g \in G$, $\varphi \in \Phi$, then the subgroup $\{g, \varphi\}$ of the group U is nilpotent as a finite p-group. Since in G there are no elements fixed with respect to Φ, it follows that Φ is not an externally locally nilpotent group of automorphisms.

This same group of automorphisms Φ of the group G is periodic, but not externally locally periodic (Lemma 1).

4. **An example proving propositions B and C.** Let G be a group which is the direct product of cyclic groups $\{a_{n,k}\}$ of order p (p is any prime number), where the first index n runs over all natural numbers, and the second index k over a complete system of residues modulo p^{n-1}. Define automorphisms φ_i $(i = 0, \pm 1, \pm 2, \dots)$ of the group G by the following equalities:

$$\varphi_i(a_{n,k}) = a_{n,k}, \quad \text{if } k \neq i \pmod{p^{n-1}};$$

$$\varphi_i(a_{n,i}) = a_{n,i}a_{n+1,i+1}.$$

We shall show that the group of automorphisms Φ, generated by all the φ_i, is externally locally nilpotent with respect to G. For this it is enough to verify that any finite set of automorphisms Σ, consisting of certain φ_i, is nilpotent. Let $\Sigma = (\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_s})$, $t = \max|i_\nu|$. Note the following facts: 1) for arbitrary $\sigma_i \in \Sigma$ one has $[a_{n,k}; \sigma_1, \sigma_2, \dots, \sigma_m] = a_{n+m,k}$ or 1; 2) if n satisfies the inequality $p^{n-1} \geq 2t + 1$, and k is the absolutely least residue modulo p^{n-1}, then for arbitrary $\sigma_i \in \Sigma$ one has $[a_{n,k}; \sigma_1, \sigma_2, \dots, \sigma_m] = a_{n+m,k+m}$ or 1. Using 1) and 2), it is now not hard to verify that for $m > \log_p(2t+1) + 2t + 1$ and any $g \in G$ one has $[g; \sigma_1, \sigma_2, \dots, \sigma_m] = 1$, i.e. Φ is an externally locally nilpotent group of automorphisms.

Now define an automorphism ψ of the group G as follows: $\psi(a_{n,k}) = a_{n,k+1}$. The automorphism ψ is a p-automorphism of the group G, and hence also a nil-automorphism (Lemma, item 2). Denote $\Gamma = \{\Phi, \psi\}$. For any integer k we have $\psi^k \varphi_i \psi^{-k} = \varphi_{i+k}$, i.e. Φ is a normal divisor of the group Γ. Since the group Γ is not externally locally nilpotent (for example, the automorphism $\psi^{-1}\varphi_0$ is not a nil-automorphism), proposition B is proved. At the same time proposition C is also proved (Lemma, item 2).

5. In this paragraph we shall construct a group of automorphisms Φ of an abelian group G with the following properties: 1) Φ is an abelian group all of whose elements are of order two; 2) Φ is an externally locally nilpotent group; 3) all automorphisms from Φ have nilpotency index $n = 2$, i.e. for any $g \in G$, $\varphi \in \Phi$ one has $[g, \varphi(2)] = 1$; 4) in the group G there are no elements fixed with respect to Φ.

As the group G we take the direct product of cyclic groups of order two $\{g_\alpha\}$, where the index α runs over all finite subsets of the set of natural numbers. Define automorphisms φ_i of the group G, where i is any natural number, by the following equalities:

$$\varphi_i(g_\alpha) = g_\alpha, \qquad \text{if } i \in \alpha;$$

$$\varphi_i(g_\alpha) = g_\alpha g_{\alpha \cup i}, \qquad \text{if } i \notin \alpha.$$

The group Φ, generated by all the φ_i, satisfies conditions 1)–4). The fulfillment of 1) and 4) is obvious, while 2) and 3) follow from the fact that if Σ is a set-

of arbitrary m elements of Φ, $g \in G$, then $[g;\ \sigma_1,\ \sigma_2, \dots, \sigma_{m+1}] = 1$, where $\sigma_i \in \Sigma$.

Remark. An analogous result can also be obtained with the aid of the well-known example of P. Hall of a p-group without center (([3]), see ([4])).

6. L. A. Kaluzhnin in ([5]) proved the theorem:

If in a group G there is an invariant series of subgroups of length n, stable with respect to its automorphism group Φ, then the group Φ is nilpotent of class at most $n - 1$.

Recently F. Hall in ([6]) (see also ([7])) showed that in the case when a Φ-stable series is not invariant but normal, Φ is also a nilpotent group of class at most $\frac{1}{2}n(n-1)$. In the same paper F. Hall gave an example showing that for $n = 3$ the exact estimate of the nilpotency class of the group Φ in these two cases is different. We shall construct an example which establishes this for any $n \geqslant 3$.

Let G be the group generated by the elements g_α, where the index α runs over all subsets of the set M consisting of the elements $1, 2, \dots, n-1$ ($n \geqslant 3$), and among these subsets we include both the empty set 0 and the set M itself. In addition, in G a system of defining relations is given: 1) $[[x,y],z]=1$, where x,y,z are arbitrary elements of G; 2) $[x,y]^2=1$, where x,y are arbitrary elements of G; 3) $[g_\alpha,g_\beta]=1$, if $\langle\alpha\rangle+\langle\beta\rangle>n$, where $\langle\alpha\rangle$ denotes the number of elements of the subset α.

Define the automorphism group $\Phi=\{\varphi_1,\varphi_2,\dots,\varphi_{n-1}\}$ of the group G by the following equalities:

$$\varphi_i(g_\alpha)=g_\alpha, \qquad \text{if } i\in\alpha;$$

$$\varphi_i(g_\alpha)=g_\alpha g_{\alpha\cup i}, \qquad \text{if } i\notin\alpha.$$

Let α be some subset of M. Denote by $h_\alpha=\prod[g_\beta,g_\gamma]$, where $\beta\cup\gamma=\alpha$, $\beta\cap\gamma=0$, and for each pair (β,γ) with such properties exactly one commutator $[g_\beta,g_\gamma]=[g_\gamma,g_\beta]$ enters the product (including also $[g_\alpha,g_0]$). Form the series of subgroups

$$G=G_0\supset G_1\supset G_2\supset\cdots\supset G_i\supset\cdots,$$

where G_i $(i=1,2,\dots)$ is generated by all elements g_α, h_α, $[g_\beta,g_\gamma]$, for which $\langle\alpha\rangle\geq i$, $\langle\beta\rangle+\langle\gamma\rangle\geq i+1$. It can be shown that this series is Φ-stable, and $G_n=1$. The group Φ, by F. Hall' s theorem, is nilpotent. Its nilpotency class is at least n, since, for example, $[\varphi_1,\varphi_2,\dots,\varphi_{n-1},\varphi_1]\neq 1$.

Let us note that the question of whether the estimate in F. Hall' s theorem is exact remains open.

7. In conclusion we give the following fact. In ([8], [9]) the following theorem is proved:

The group of locally stable automorphisms Φ of a group G is generalized periodic if and only if $[G,\Phi]$ is a periodic group.

For the opposite case, in ([8]) it is proved that the group of locally stable automorphisms Φ of the group G will be complete if $[G,\Phi]$ is a group without torsion. However, the converse theorem is no longer true here, as the following example shows.

Take a group G which is the direct product of cyclic groups:

$$G=\{g_1\}\times\{g_2\}\times\{g_3\}\times\{g_4\}\times\{h\},$$

where the group $\{h\}$ is of order two, and the remaining ones are infinite. Define the automorphism group $\Phi=\{\varphi_1,\varphi_2\}$ of the group G by the following equalities:

$$\varphi_1(g_1) = g_1 g_2; \quad \varphi_1(g_2) = g_2 g_3; \quad \varphi_1(g_3) = g_3; \quad \varphi_1(g_4) = g_4; \quad \varphi_1(h) = h;$$

$$\varphi_2(g_1) = g_1 g_2^{-1} h; \quad \varphi_2(g_2) = g_2 g_4; \quad \varphi_2(g_3) = g_3; \quad \varphi_2(g_4) = g_4; \quad \varphi_2(h) = h.$$

The group Φ, as one can verify, is stable and pure, while $[G, \Phi]$ contains a nontrivial periodic part.

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Note: Figure translations are in progress. See original paper for figures.