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Abstract

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ON CONJUGATE FUNCTIONS AND SINGULAR CAUCHY INTEGRALS

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Let E denote a linear family of functions defined on the interval $[0, 2\pi]$, containing all summable functions $f(x)$ and their conjugates

$$\overline{f}(x) = -\frac{1}{\pi}\int_0^{2\pi} f(t)\,\frac{1}{2}\operatorname{ctg}\frac{1}{2}(t-x)\,dt, \qquad 0 \le x \le 2\pi.$$

Let, further, Φ be an arbitrary linear functional defined on E and satisfying the condition: if $f \in L(0, 2\pi)$, then

$$\Phi(f) = \int_1^{2\pi} f\,dx, \qquad \Phi(\overline{f}) = 0.$$

We shall call the number $\Phi(f)$ the Φ-integral of the function $f \in E$, and introduce for it the notation

$$\Phi(\overline{f}) = (\Phi)\int_0^{2\pi} f\,dx.$$

Examples of Φ-integrals are the A-integral ([1,2]) and the B-integral ([3,4]). In papers ([2,5–7]), with the aid of the A-integral, a number of facts concerning conjugate functions and integrals of Cauchy type were established. In the present note it is shown that the results obtained in the works cited are consequences not of the specific properties of the A-integral, but only of the fact of the existence of an integral of conjugate functions. Namely, it is proved that the indicated results are valid also for any Φ-integral. In addition, some properties of the singular Cauchy integral are established in dependence on the properties of the line of integration.

1. Theorem 1. Let $f \in L(0, 2\pi)$, and let the 2π-periodic function φ satisfy the Dini condition: $\omega(\sigma; \varphi)\sigma^{-1} \in L(0, 2\pi)$, where $\omega(\sigma; \varphi)$ is the modulus of continuity of the function φ. Then $\varphi\overline{f} = f_1 + f_2$, where $f_1, f_2 \in L(0, 2\pi)$, and

$$(\Phi) \int_0^{2\pi} \varphi(x)\overline{f}(x)\, dx = -\int_0^{2\pi} \overline{\varphi}(x) f(x)\, dx\ ^*. \tag{1}$$

Indeed,

$$\varphi(x)\overline{f}(x) = \overline{[\varphi(x) f(x)]} - \frac{1}{\pi}\int_0^{2\pi} f(t)\, \frac{\varphi(x)-\varphi(t)}{2\,\mathrm{tg}\,\frac{1}{2}(t-x)}\, dt,$$

where the function

$$f(t)\, \frac{\varphi(x)-\varphi(t)}{2\,\mathrm{tg}\,\frac{1}{2}(t-x)}$$

is, as is easily verified, summable on the square $[0, 2\pi;\, 0, 2\pi]$. Taking the Φ-integral of both sides of the last equality and

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* If the sign (Φ) is absent before the integral, the Lebesgue integral is meant.

then changing the order of integration in the resulting repeated integral, we obtain equality (1).

In the case of the A-integral, under somewhat more general assumptions on φ, equality (1) is proved in ([2]) by a method different from that given above. From equality (1), just as in the case of the A-integral ([2,5]), the following consequences follow:

Corollary 1. The series conjugate to the Fourier series for $f \in L(0, 2\pi)$ is the Fourier series for $\bar{f}$ in the sense of the Φ-integral.

Corollary 2. If $f \in L(0, 2\pi)$, and $u(r, \vartheta)$ is its Poisson integral, then the function $v(r, \vartheta)$, harmonically conjugate to $u(r, \vartheta)$, is representable in the form of the Φ-Poisson integral

$$v(r, \vartheta) = \frac{1}{2\pi}(\Phi)\int_0^{2\pi} \bar{f}(t)\, \frac{1-r^2}{1-2r\cos(t-\vartheta)+r^2}\, dt.$$

2. Suppose that a rectifiable Jordan contour Γ is given in the complex plane. Its equation can be written in the form $t = t(s)$, $0 \le s \le \gamma$, where s is the arc abscissa and γ is the length of the contour. We shall call Γ a D-contour if the function $t'(s)$ satisfies the Dini condition (the class of D-contours includes Lyapunov contours).

Lemma. If Γ is a simple closed D-contour of length 2π, and $f(t) \in L(\Gamma)$, then for almost all $s_0 \in [0, 2\pi]$

$$\lim_{\varepsilon\to 0}\left(\int_0^{s_0-\varepsilon}+\int_{s_0+\varepsilon}^{2\pi}\right)\frac{f[t(s)]\,t'(s)}{t(s)-t(s_0)}\,ds=\int_0^{2\pi} f[t(s)]\,\frac{1}{2}\operatorname{ctg}\frac{1}{2}(s-s_0)\,ds+\int_0^{2\pi} f[t(s)]K(s,s_0)\,ds, \tag{2}$$

where the function

$$K(s,s_0)\equiv f[t(s)]\left[\frac{t'(s)}{t(s)-t(s_0)}-\frac{1}{2}\operatorname{ctg}\frac{1}{2}(s-s_0)\right]$$

is summable on the square $[0,2\pi;\ 0,2\pi]$.

An analogous proposition in the case of an open Lyapunov contour was proved in ([8]).

It follows from the lemma that, if Γ is a D-contour and $f(t)\in L(\Gamma)$, then for almost all $t_0\in\Gamma$ the function

$$S(f;t_0)=\frac{1}{\pi i}\int_\Gamma\frac{f(t)\,dt}{t-t_0},$$

is defined, where the integral is understood in the sense of the Cauchy principal value (the latter integral is called a singular Cauchy integral). Moreover, with the aid of the lemma, certain results obtained for conjugate functions, for example the inequalities of Riesz, Kolmogorov, and Zygmund, and the Riesz equality, can also be proved for the function $S(f;t)$ given on a D-contour. The proofs can be carried out in the same way as in ([8]), where, in the case of a Lyapunov contour, the Riesz inequality and equality were proved.

For complex functions given on Γ, one can introduce the notion of a Φ-integral in the same way as was done in ([6]) in the case of the A-integral. Denote by $E(\Gamma)$ the class of Φ-integrable functions on Γ. With the aid of equalities (1), (2), and Fubini’ s theorem on the interchange of the order of integration, the following theorem is easily proved:

Theorem 2. Let Γ be a finite collection of simple closed mutually nonintersecting D-contours, $f(t)\in L(\Gamma)$, and let $\varphi(t)$ satisfy

Dini condition. Then $\varphi(t)S(f;t)=f_1(t)+S(f_2;t)$, where $f_1(t),f_2(t)\in L(\Gamma)$; $\varphi(t)S(f;t)\in E(\Gamma)$, and

$$(\Phi)\int_\Gamma\varphi(t)S(f;t)\,dt=-\int_\Gamma S(\varphi;t)f(t)\,dt.$$

3. Consider the Cauchy-Lebesgue type integral

$$F(z)=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)\,dt}{t-z}, \qquad z\notin\Gamma,$$

where Γ is a D-contour and $f(t)\in L(\Gamma)$. From the theorem of I. I. Privalov ([9]) and equality (2) it follows that for almost all $t\in\Gamma$ there exist angular boundary values of the function $F(z)$, which are expressed by the Sokhotski-Plemelj formulas and, consequently, possess the same integrability properties as the function $S(f;t)$.

With the help of Theorem 2 one establishes certain results concerning the representation of analytic functions by a Cauchy-type integral and by the Cauchy integral. In particular, the following theorem follows from Theorem 2:

Theorem 3. Let the line Γ satisfy the conditions of Theorem 2 and bound a connected domain G. Then the analytic function $F(z)$, representable in G by a Cauchy-Lebesgue type integral, is representable also by a Φ-Cauchy integral.

Indeed, let $F(t)$ be the angular boundary values of the function $F(z)$. Then, according to Theorem 2 and the Sokhotski-Plemelj formula, if $z\notin\Gamma$, then

$$\begin{aligned}
\frac{1}{2\pi i}(\Phi)\int_\Gamma \frac{F(t)\,dt}{t-z} &= \frac{1}{4\pi i}\int_\Gamma \frac{f(t)\,dt}{t-z}+\frac{1}{(2\pi i)^2}(\Phi)\int_\Gamma \frac{dt}{t-z}\int_\Gamma \frac{f(\tau)\,d\tau}{\tau-t}\\
&= \frac{1}{4\pi i}\int_\Gamma \frac{f(t)\,dt}{t-z}+\frac{1}{(2\pi i)^2}\int_\Gamma f(\tau)\,d\tau\int_\Gamma \frac{dt}{(t-z)(\tau-t)}\\
&= \frac{1}{4\pi i}\int_\Gamma \frac{f(t)\,dt}{t-z}+\frac{1}{(2\pi i)^2}\int_\Gamma \frac{f(\tau)\,d\tau}{\tau-z}\int_\Gamma \frac{dt}{t-z}+\frac{1}{(2\pi i)^2}\int_\Gamma \frac{f(\tau)\,d\tau}{\tau-z}\int_\Gamma \frac{dt}{\tau-t}\\
&= \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f(t)\,dt}{t-z}, & \text{for } z\in G,\\[2ex] 0, & \text{for } z\in \overline{G}+\Gamma. \end{cases}
\end{aligned}$$

Theorem 3 in the case of the A-integral and a simply connected domain bounded by a Lyapunov contour was proved by P. L. Ul' yanov ([5–7]) with the aid of the theory of conformal mappings.

It follows from Theorem 3 that, for representability of a function $F(z)$ in G by a Cauchy-Lebesgue integral, summability of the boundary function $F(t)$ is necessary and sufficient. The latter assertion in the case when G is a circle was proved by V. I. Smirnov (see, for example, ([9])). In the case of a simply connected domain bounded by a Lyapunov contour, it follows from the theorem of P. L. Ul' yanov.

4. In the case when the line of integration is open, and also when it is closed but has corner points, the function $S(f;t)$, generally speaking, is not integrable in the sense of any Φ-integral satisfying additionally certain natural requirements.

Example 1. Let Γ be the rectilinear segment $[0, 2\pi]$, and let $f(t) = f[t(s)] = 1/s|\ln s|^2$ for $0 < s \leq \frac{1}{2}$, and $f(t) = f[t(s)] = 0$ at the remaining points of $[0, 2\pi]$.

Then, if the Φ-integral is a positive functional, i.e. $\Phi(f) \geq 0$ if $f \geq 0$ (such are, for example, the A-integral and the B-integral), the function $S(f;t)$ is not integrable on Γ in the sense of the Φ-integral.

Example 2. Let Γ be the boundary of the square $[0, \pi/2; 0, \pi/2]$, and let $f(t) = f[\,l(s)\,]$ be the same function as in Example 1, considered as a function of arc length.

Then, if the Φ-integral is a positive functional and satisfies the condition: if a function is Φ-integrable on the intervals $[a, a+b/2]$ and $[a+b/2, b]$, then it is also Φ-integrable on $[a, b]$* (this condition is satisfied, for example, by the A-integral), the function $S(f;t)$ is not integrable on Γ in the sense of the Φ-integral.

We also note that the function $S(f;t)$ constructed in Example 2 is not integrable in the sense of the B-integral either.

Thus, the limiting functions of an integral of Cauchy type for an open contour or for a closed contour with corner points are, generally speaking, not integrable in the sense of any Φ-integral satisfying the conditions indicated in Examples 1 and 2. Consequently, Theorem 3, generally speaking, will not be valid for any such Φ-integral if contours with corner points are taken as the line of integration.

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$$* \text{ By } (\Phi)\int_a^b f\,dx \text{ we mean } (\Phi)\int_0^{2\pi} \chi f\,dx,$$

where χ is the characteristic function of the interval (a, b).

Note: Figure translations are in progress. See original paper for figures.