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ON PARACOMPACT AND FINALLY COMPACT SPACES*

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Abstract

Full Text

MATHEMATICS

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ON PARACOMPACT AND FINALLY COMPACT SPACES*

(Presented by Academician P. S. Aleksandrov, 28 VI 1961)

In this note the following new characterizations of paracompact and finally compact** spaces are given.

Theorem 1. *In order that a Hausdorff space X be paracompact, it is necessary and sufficient that for every open cover of this space there exist an ω -mapping*** $f : X \rightarrow Y$ onto some metric space Y .*

Theorem 2. *In order that a regular space X be finally compact, it is necessary and sufficient that for every open cover of this space there exist an ω -mapping $f : X \rightarrow Y$ onto some metric space Y with a countable base.*

The sufficiency of our conditions follows from the following proposition, true without any restrictive assumptions whatever concerning the space X :

I. *If for every open cover ω of a space X there exists an ω -mapping $f : X \rightarrow Y$ onto some paracompact (respectively, finally compact) space Y , then X itself is paracompact (respectively, finally compact).*

Proof. Let ω be an arbitrary open cover of the space X . By assumption there exists an ω -mapping f of the space X onto some paracompact (respectively, finally compact) space Y . For each point $y \in Y$ take a neighborhood Oy such that its inverse image $f^{-1}Oy$ is contained in some $U \in \omega$. These Oy form some cover β' of the space Y ; inscribe in it a locally finite (respectively, countable) cover $\beta\{V\}$. Then $\alpha = \{f^{-1}V\}$ is a cover of the space X , inscribed in the cover ω . If β is countable, then so is α . If β is locally finite, then so is α . Indeed, let $x \in X$ be arbitrary and let a neighborhood Oy of the point $y = fx$ meet only finitely many elements V of the cover β . Then $f^{-1}Oy$ is a neighborhood of the point x , meeting only finitely many elements $f^{-1}V$ of the cover α . Proposition I is proved.

In the same way one proves the proposition:

II. *If for every open cover ω of a space X there exists an ω -mapping of this space onto a strongly (respectively,*

* *Note added in proof, 11 X 1961.* After the text of this note had already been set, the author learned that the main result (Theorem I) had been formulated and proved by M. Katetov in the appendix to E. Čech's book *Topological Spaces*.

** A space X is called finally compact (or Lindelöf) if in every open cover of this space one can inscribe a countable (or finite) open cover.

*** Let a cover ω of a space X be given; a continuous mapping f of the space X into some space Y is called an ω -mapping if, for every point $y \in Y$, there exists a neighborhood Oy whose inverse image $f^{-1}Oy$ is contained in at least one element of the cover ω .

to a weakly, i.e., pointwise, paracompact space Y , then the space X itself is strongly (respectively weakly) paracompact.

We now prove the proposition:

III. Let X be paracompact. For every open covering ω of the space X there exists an ω -mapping $f : X \rightarrow Y$ onto some metric space Y . If, moreover, X is finally compact, then Y may be assumed to be a metric space with a countable base.

Remark. Proposition III is contained in results of C. H. Dowker (¹), proved by him on the basis of quite different considerations.

Proof. If X is paracompact, then, by the well-known Stone theorem, in every open covering ω one can star-refine some open covering ω_1 . If X is a finally compact (regular) space, then it is, as is known, paracompact (see, for example, (²)), and then in every covering ω one can star-refine some countable covering ω_1 . Thus, for a paracompact X and an arbitrary covering ω of it we can construct a sequence of coverings

$$\omega = \omega_0, \omega_1, \dots, \omega_k, \dots,$$

where each ω_k , $k = 1, 2, \dots$, is star-refined in ω_{k-1} ; moreover, in the case of a finally compact space X , it may be assumed that all ω_k , $k \geq 1$, are countable coverings.

We construct in X a function $d(x, y)$ of two points (a "generalized pseudometric") as follows:

1°. If for two given points x, y there exists no element of the covering ω_0 containing both these points, we put $d(x, y) = 2$.

2°. Suppose that case 1° does not occur; then there exists such a $k \geq 0$ that both points x and y are contained in some element of the covering ω_k . If among these k there is a greatest one, denoting it by $k(x, y)$, put

$$d(x, y) = 1/2^{k(x, y)}.$$

If there is no such greatest k , put $d(x, y) = 0$. The function $d(x, y)$ is, obviously, symmetric and, besides the value 0, assumes only values of the form $1/2^k$, $k =$

$-1, 0, 1, 2, \dots$. The following property of the function $d(x, y)$ is easily proved: if

$$d(x, y) \leq 1/2^{k+1}, \quad d(y, z) \leq 1/2^{k+1},$$

then

$$d(x, z) \leq 1/2^k.$$

From this, in particular, it follows: if $d(x, y) = 0$, $d(y, z) = 0$, then also $d(x, z) = 0$. Therefore the whole space X splits into pairwise disjoint classes consisting of points whose pairwise “distance” $d(x, y)$ is equal to zero. Denote by Ξ the set of these classes. In each class $\xi \in \Xi$ choose a point $x_\xi \in \xi$, and for two classes put

$$d(\xi, \eta) = d(x_\xi, x_\eta).$$

Symmetry is obviously satisfied; $d(\xi, \eta) = 0$ if and only if $\xi = \eta$. If

$$d(\xi, \eta) \leq 1/2^{k+1}, \quad d(\eta, \zeta) \leq 1/2^{k+1},$$

then this means that

$$d(x_\xi, x_\eta) \leq 1/2^{k+1}, \quad d(x_\eta, x_\zeta) \leq 1/2^{k+1},$$

but then

$$d(x_\xi, x_\zeta) \leq 1/2^k,$$

i.e.,

$$d(\xi, \zeta) \leq 1/2^k.$$

Thus, the set Ξ , with “distance” $d(\xi, \eta)$, is a generalized metric space, and consequently, by Chittenden’s theorem,* a metrizable space: we can introduce in Ξ a genuine metric $\rho(\xi, \eta)$, topologically equivalent to the “metric” $d(\xi, \eta)$.

We assign to each point $x \in X$ the class $\xi \in \Xi$ containing it. We obtain a mapping $f : X \rightarrow \Xi$. This mapping is continuous. Indeed, let $fx_0 = \xi_0$ and let an arbitrary

$$\varepsilon = 1/2^k$$

be given. Take

$$U^{k+2} \in \omega_{k+2}$$

containing the point x_0 ; it suffices to prove that for $x \in U^{k+2}$, $fx = \xi$, we have

$$d(\xi_0, \xi) \leq 1/2^k.$$

Choose x_{ξ_0}, x_ξ . Then, by definition,

$$d(\xi_0, \xi) = d(x_{\xi_0}, x_\xi),$$

so that it is required to prove the inequality

$$d(x_{\xi_0}, x_\xi) \leq 1/2^k.$$

By construction,

$$d(x_{\xi_0}, x_0) = 0 < 1/2^{k+2}, \quad d(x_0, x) \leq 1/2^{k+2},$$

hence

$$d(x_{\xi_0}, x) \leq 1/2^{k+1};$$

also

$$d(x, x_{\xi}) = 0 < 1/2^{k+1},$$

hence

$$d(x_{\xi_0}, x_{\xi}) \leq 1/2^k,$$

which proves the continuity of the mapping f . We prove that f is an ω -mapping. Let $f x_0 = \xi_0$, $x_0 \in U^1 \in \omega_1$. It is necessary to prove that

$$f^{-1}O_d(\xi_0, 1/2^4)$$

* The proof of Chittenden's theorem is reproduced in ⁽³⁾, pp. 966-972.

is contained in some $U \in \omega_0$. Let $x \in f^{-1}O_d(\xi_0, 1/2^4)$. It is enough to show that x is contained in the star of the point x_0 with respect to ω_1 , i.e. that $d(x_0, x) \leq 1/2$. But since $x \in \xi$, $x \in f^{-1}O_d(\xi_0, 1/2^4)^*$, we have

$$d(\xi_0, \xi) = d(x_{\xi_0}, x_{\xi}) < 1/2^4; \tag{1}$$

$$d(x_0, x_{\xi_0}) = 0 \leq 1/2^4, \quad d(x, x_{\xi}) = 0. \tag{2}$$

From (1) and (2) it follows that $d(x_0, x_{\xi}) \leq 1/2^3$. But $d(x_{\xi}, x) = 0 \leq 1/2^3$, and therefore $d(x_0, x) \leq 1/2^2 < 1/2$, as was required.

Let us return to the generalized pseudometric $d(x, y)$ constructed in the space X . We have $O_d(x, 1/2^k) = \mathcal{E}(x', d(x', x) \leq 1/2^{k+1})$, since $O_d(x, 1/2^k)$ consists of all points x' for which there exists an element $U^{k'} \in \omega_{k'}$, $k' \geq k+1$, containing both points x and x' , which simply means that $O_d(x, 1/2^k)$ coincides with the star of the point x in the covering ω_{k+1} . Therefore all balls $O_d(x, 1/2^k)$ are open sets in X . Taking them as a base of a certain (coarser) topology, we turn the set of all points of the space X into a neighborhood pseudometric space X' ; moreover (by virtue of the openness of the balls $O_d(x, 1/2^k)$ in the topology X) the identity mapping of X onto X' is continuous. The mapping $f : X \rightarrow \Xi$, considered as a mapping of the pseudometric space X' onto the generalized-metric space Ξ , is continuous. Therefore under this mapping f every set D everywhere dense in X' goes over into a set fD everywhere dense in the metrizable space Ξ . If the space X is finally compact, then all coverings ω_k , $k \geq 1$, may be assumed countable; choosing one point in each element of each covering ω_k , we obtain a

countable set D , everywhere dense in X' ; then fD is a countable everywhere dense set in the metrizable space Ξ , whereby assertion III is proved.

Theorems 1 and 2 may also be given the following form:

A Hausdorff space X is paracompact if and only if for every covering ω a pseudometric $d(x, y)$ can be constructed satisfying the following conditions:

- a) *the balls $O_d(x, \varepsilon)$ of this pseudometric are open in X ;*
- b) *for every point $x \in X$ there is a spherical neighborhood $O_d(x, \varepsilon)$ contained in some element of the covering ω .*

At the same time, finally compact spaces are distinguished among all paracompact spaces by the additional requirement that each of these pseudometrics be separable (i.e. transform X into a pseudometric space with a countable everywhere dense set).

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CITED LITERATURE

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2. Yu. M. Smirnov, Izv. AN SSSR, ser. matem., **20**, 253 (1956).
3. P. S. Uryson, *Tr. po topologii i drugim oblastyam matematiki*, **2**, Moscow-Leningrad, 1951.

* By $O_d(\xi_0, 1/2^4) = \mathcal{E}(\xi, d(\xi, \xi_0) < 1/2^4)$ we denote a ball in the sense of the pseudometric $d(\xi, \eta)$. Similarly below.

Note: Figure translations are in progress. See original paper for figures.

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