ON THE CLASSIFICATION OF FIBERED SPACES

1961

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Abstract

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MATHEMATICS

A. L. ONISHCHIK

ON THE CLASSIFICATION OF FIBERED SPACES

(Presented by Academician P. S. Aleksandrov on 17 VII 1961)

The well-known theorems of de Rham and Dolbeault reduce the computation of cohomology with real coefficients and of cohomology with coefficients in the sheaf of germs of analytic functions to certain problems of the calculus of differential forms on a manifold. In the present paper it is shown that the same method is applicable to the study of cohomology with coefficients in certain non-Abelian sheaves. It turns out that the problem of computing these cohomologies (or, equivalently, the problem of classifying fibered spaces) can also, in a known sense, be reduced to a certain problem of the calculus of differential forms.

1. Let X be a smooth manifold, and $\mathfrak{G}$ a Lie group. Denote by $D(X, \mathfrak{G})$ the group of all smooth mappings $X \to \mathfrak{G}$, by G the Lie algebra of the group $\mathfrak{G}$, and by $\mathcal{L}^p(X, G)$ the vector space of differential forms of degree p on X with values in G. We shall now consider certain operations on these differential forms, studied by Aranyos ([1]).

To each mapping $f \in D(X, \mathfrak{G})$ assign the differential form $\delta f = f^{-1} df \in \mathcal{L}^1(X, G)$. For $\alpha \in \mathcal{L}^1(X, G)$ put

$$C(f)\alpha = f\alpha f^{-1} + \delta(f^{-1}).$$

It is verified that

$$C(f_1 f_2) = C(f_1)C(f_2)$$

and that for the constant mapping $\varepsilon(x) = e$ we have

$$C(\varepsilon)\alpha = \alpha.$$

Thus an action of the group $D(X, \mathfrak{G})$ on $\mathcal{L}^1(X, G)$ is defined. Next, consider the mapping $\nabla : \mathcal{L}^1(X, G) \to \mathcal{L}^2(X, G)$, defined by the formula

$$\nabla\alpha = d\alpha + \tfrac{1}{2}[\alpha, \alpha] \quad (\alpha \in \mathcal{L}^1(X, G)),$$

where the square brackets denote the commutation of differential forms. The operators δ and ∇ play, in the theory of differential forms with values in G, a role analogous to that of the ordinary operators of exterior differentiation. In particular, consider the equation

$$\delta f = \alpha \tag{1}$$

and determine the conditions for its solvability with respect to f, for a given $\alpha \in \mathcal{L}^1(X, G)$.

Lemma 1. For every $f \in D(X, \mathfrak{G})$ we have $\nabla \delta f = 0$. If $\nabla \alpha = 0$, then equation (1) has a solution in a neighborhood of every point of the manifold X.

For the proof of the lemma it is enough to verify that the equality $\nabla \alpha = 0$ is the condition of complete integrability of the system of partial differential equations which is obtained from equation (1) upon passing to coordinates.

Consider now the case where X is a complex manifold and $\mathfrak{G}$ is a complex Lie group. Denote by $\mathcal{L}^{p,q}(X, G)$ the space of differential forms of type (p, q) on X with values in G. Introduce the following notation:

$$\delta'' f = f^{-1} d'' f,$$

$$C''(f)a = faf^{-1} + \delta''(f^{-1}) \quad (f \in D(X, \mathfrak{G}), \quad a \in \mathcal{L}^{0,1}(X, G)),$$

$$\nabla'' a = d'' a + \frac{1}{2}[a, a].$$

It is readily verified that the operators $C''(f)$ define an action of the group $D(X, \mathfrak{G})$ on $\mathcal{L}^{0,1}(X, G)$.

Lemma 2. For every $f \in D(X, \mathfrak{G})$ we have $\nabla'' \delta'' f = 0$. If $\nabla'' a = 0$, then the equation

$$\delta'' f = a \tag{2}$$

has a solution in a neighborhood of each point of the manifold X.

Proof. It is verified that the equality $\nabla'' a = 0$ is the complete integrability condition for equation (2). One must then apply the "complex Frobenius theorem," proved by Nirenberg ([2]).

2. We now pass to the question of the classification of complex-analytic fibered spaces. Let X be a complex manifold, $\mathfrak{G}$ a complex Lie group. Suppose that an analytic fiber bundle $\mathfrak{F}$ with fiber $\mathfrak{G}$ and base X is given. The bundle $\mathfrak{F}$ is specified by a coordinate covering $\{U_i\}$ of the manifold X and by transition functions

f_{ij}, defined in $U_i \cap U_j$. Our assumptions mean that the f_{ij} are automorphisms of the group $\mathfrak{G}$, and moreover the mappings defined by them $(U_i \cap U_j) \times \mathfrak{G} \to \mathfrak{G}$ are analytic. We shall consider analytic $\mathfrak{F}$-principal fibered spaces with base X; these fibered spaces are in a natural correspondence with the elements of the cohomology set $H^1(X, \mathfrak{A}(\mathfrak{F}))$, where $\mathfrak{A}(\mathfrak{F})$ is the sheaf of germs of analytic sections of the bundle $\mathfrak{F}$ (see ([3]) or ([4])).

Let $\dot{f}_{ij}$ be the automorphism of the Lie algebra G of the group $\mathfrak{G}$ corresponding to the automorphism f_{ij}. The transition functions $\dot{f}_{ij}$ define over X an analytic bundle F of Lie algebras with fiber G. Denote by $\Omega^{p,q}(F)$ the sheaf of germs of differential forms of type (p,q) with values in F. Also denote by $\mathfrak{D}(\mathfrak{F})$ the sheaf of germs of differentiable sections of the bundle $\mathfrak{F}$, and by $A(\mathfrak{F}), D(\mathfrak{F}), \mathcal{L}^{p,q}(F)$ the groups of continuous sections of the sheaves $\mathfrak{A}(\mathfrak{F}), \mathfrak{D}(\mathfrak{F}), \Omega^{p,q}(F)$. The operators defined in §1 give rise to homomorphisms of sheaves of sets

$$\delta'' : \mathfrak{D}(\mathfrak{F}) \to \Omega^{0,1}(F),$$

$$\nabla'' : \Omega^{0,1}(F) \to \Omega^{0,2}(F).$$

Denote by $\mathfrak{Z}''(F)$ the subsheaf of $\Omega^{0,1}(F)$ consisting of those a for which $\nabla'' a = 0$, and by $Z''(F)$ the set of continuous sections of the sheaf $\mathfrak{Z}''(F)$. The group $D(\mathfrak{F})$ acts on $\mathcal{L}^{0,1}(F)$, and $Z''(F)$ is carried into itself.

Theorem 1. The set of analytic $\mathfrak{F}$-principal fibered spaces, differentiably equivalent to $\mathfrak{F}$, is in one-to-one correspondence with the orbits of the group $D(\mathfrak{F})$ in the set $Z''(F)$.

Proof. From Lemma 2 it is clear that the sheaf $\mathfrak{D}(\mathfrak{F})$ acts transitively on the sheaf of sets $\mathfrak{Z}''(F)$, while the sheaf of stationary subgroups of the zero section coincides with $\mathfrak{A}(\mathfrak{F})$. In other words, there is an exact sequence of sheaves:

$$\varepsilon \to \mathfrak{A}(\mathfrak{F}) \to \mathfrak{D}(\mathfrak{F}) \to \mathfrak{Z}''(F) \to 0.$$

From it follows the exact cohomology sequence ([3,4]).

$$\varepsilon \to A(\mathfrak{F}) \to D(\mathfrak{F}) \to Z''(F) \xrightarrow{\delta} H^1(X, \mathfrak{A}(\mathfrak{F})) \to H^1(X, \mathfrak{D}(\mathfrak{F})).$$

By exactness, $Z''(F)$ is mapped precisely onto the set of fiber spaces in question in the statement of the theorem. Moreover, $\delta\alpha = \delta\beta$ if and only if α is carried into β by an element of the group $D(\mathfrak{F})$. Thus the theorem is proved.

Let us now pass to the more general question of the classification of all $\mathfrak{F}$-principal analytic fiber spaces with base X, differentiably equivalent to a certain (generally speaking, nontrivial) analytic $\mathfrak{F}$-principal fiber space P. To each element of the group $\mathfrak{G}$ there corresponds an inner automorphism of the group generated by this element. This correspondence allows one to define an action of the bundle of groups $\mathfrak{F}$ on itself. Consequently, to the $\mathfrak{F}$-principal bundle P there corresponds a new bundle of groups with fiber $\mathfrak{G}$, which we shall denote by $\mathfrak{F}_P$. Let Q be another $\mathfrak{F}$-principal fiber space. Consider the bundle $Q \vee P$ with base X, obtained from Q and P as a result of the direct product of the fibers over each point of the base; this bundle is $(\mathfrak{F} \vee \mathfrak{F})$-principal. Since $\mathfrak{F} \vee \mathfrak{F}$ acts on $\mathfrak{F}$ by means of left and right translations, the bundle $Q \vee P$ defines a certain bundle with fiber $\mathfrak{G}$, which is denoted by $Q \vee_{\mathfrak{F}} P$. It is known that the bundle $Q \vee_{\mathfrak{F}} P$ is $\mathfrak{F}_P$-principal [3,4]. Thus, every $\mathfrak{F}$-principal bundle P defines a mapping

$$\alpha_P^* : H^1(X, \mathfrak{A}(\mathfrak{F})) \to H^1(X, \mathfrak{A}(\mathfrak{F}_P)).$$

Obviously, $\alpha_P(P) = \mathfrak{F}_P$. It is known that α_P is a one-to-one correspondence [3,4].

It is also easy to prove the following lemma:

Lemma 3. α_P maps the set of all $\mathfrak{F}$-principal fiber spaces differentiably equivalent to the bundle P onto the set of all differentiably trivial $\mathfrak{F}_P$-principal fiber spaces.

From Lemma 3 and Theorem 1 there follows the following

Theorem 2. Let P be an $\mathfrak{F}$-principal analytic fiber space. Denote by F_P the bundle of Lie algebras associated with the bundle of groups $\mathfrak{F}_P$ defined above. Then principal analytic fiber spaces differentiably equivalent to the bundle P are in one-to-one correspondence with the orbits of the group $D(\mathfrak{F}_P)$ in the set $Z''(F_P)$.

It is useful to indicate an explicit construction which makes it possible to associate with an $\mathfrak{F}$-principal bundle Q, differentiably equivalent to the bundle P, a differential form with values in F_P which is contained in the orbit of the group $D(\mathfrak{F}_P)$ corresponding to Q. Let $\{U_i\}$ be a coordinate covering of the manifold X for the bundles P and Q, and let P and Q be given in this covering by cocycles $\{p_{ij}\}$ and $\{q_{ij}\}$ with coefficients in $\mathfrak{A}(\mathfrak{F})$. Since P and Q are differentiably equivalent, for a suitable choice of the covering $\{U_i\}$ there exists a 0-cochain $\{f_i\}$ with coefficients in $\mathfrak{D}(\mathfrak{F})$ such that $q_{ij} = f_i p_{ij} f_j^{-1}$ in $U_i \cap U_j$. Define in U_i a form φ_i with values in F by the equality $\varphi_i = \delta'' f_i$. It is easy to show that $\varphi_i = p_{ij} \varphi_j p_{ij}^{-1}$ in $U_i \cap U_j$. Consequently, the φ_i define a differential form φ with values in F_P on X. This is the desired form.

3. Theorems 1 and 2 proved above may be regarded as a generalization of Dolbeault' s theorem. We shall now indicate an analogous "generalization of de Rham' s theorem." Let X be a real smooth manifold, $\mathfrak{G}$ a real

Lie group. A mapping of some subset $W \subset X$ into some set is called **locally constant** if it is constant on each connected component of the set W. A fibered space with base X is called **locally constant** if its transition functions are locally constant. Suppose that a certain locally constant fibration by groups $\mathfrak{F}$ with base X and fiber $\mathfrak{G}$ is given. Fix some locally constant $\mathfrak{F}$-principal fibration P, and set ourselves the task of describing the set of all $\mathfrak{F}$-principal locally constant fibrations differentiably equivalent to P. With the fibration P there is associated a new locally constant fibration by groups $\mathfrak{F}_P$; denote by F_P the corresponding locally constant fibration by Lie algebras. Let $D(\mathfrak{F}_P)$ be the group of all differentiable sections of the fibration $\mathfrak{F}_P$, and let $\mathcal{L}^p(F_P)$ be the space of differential forms of degree p on X with values in F_P. Mappings are defined

$$\delta: \quad D(\mathfrak{F}_P) \to \mathcal{L}^1(F_P),$$

$$\nabla: \quad \mathcal{L}^1(F_P) \to \mathcal{L}^2(F_P).$$

The group $D(\mathfrak{F}_P)$ acts on $\mathcal{L}^1(F_P)$ by the formula $C(f)a = faf^{-1} + \delta(f^{-1})$. Denote by $Z(F_P)$ the set of those forms $a \in \mathcal{L}^2(F_P)$ for which $\nabla a = 0$. The set $Z(F_P)$ is invariant with respect to $D(\mathfrak{F}_P)$. Arguments analogous to the arguments of §2 lead to the following theorem:

Theorem 3. *Locally constant $\mathfrak{F}$-principal fibered spaces differentiably equivalent to the fibration P are in one-to-one correspondence with the orbits of the group $D(\mathfrak{F}_P)$ in the set $Z(F_P)$.*

Moscow State University
named after M. V. Lomonosov

Received
11 VII 1961

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Note: Figure translations are in progress. See original paper for figures.