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Abstract

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C'-CONNECTIONS IN HOMOGENEOUS SPACES AND THEIR TOTALLY GEODESIC SUBMANIFOLDS

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A subgroup g of a Lie group G is called framed (([1], cf. [2]) if the Lie algebra $\dot{G}$ of the group G is decomposed into a direct sum of subspaces $\dot{g}$ and H, where $\dot{g}$ is the subalgebra corresponding to the subgroup g, and H is invariant with respect to the corresponding $\dot{g}$ subgroup $\dot{g}^*$ of the group G^* attached to G.

Let the right-invariant basic differential forms of the group G be divided into $q+2$ groups $\omega^{i_\lambda}, \omega^i$ $(\lambda = 0, 1, \dots, q)$ in such a way that the structure equations of the group take the form

$$D\omega^{i_\lambda} = C^{i_\lambda}_{k_\lambda l}[\omega^{k_\lambda}\omega^l] + C^{i_\lambda}_{k_\lambda l_\nu}[\omega^{k_\lambda}\omega^{l_\nu}] + \tfrac{1}{2}C^{i_\lambda}_{k_\mu l_\mu}[\omega^{k_\mu}\omega^{l_\mu}],$$
$$D\omega^i = \tfrac{1}{2}C^i_{kl}[\omega^k\omega^l] + \tfrac{1}{2}C^i_{k_\lambda l_\lambda}[\omega^{k_\lambda}\omega^{l_\lambda}], \tag{1}$$

where $\nu > \lambda,\ \mu < \lambda,\ \lambda = 0, 1, \dots, q$.

This is equivalent (see [1], item 5) to specifying in G a series of subgroups $g_\lambda,\ g$ $(\lambda = 1, \dots, q;\ g_1 \supset g_2 \supset \dots \supset g_q \supset g)$, framed by the subspaces H_λ, H, with $H \supset H_q \supset \dots \supset H_1$.

Suppose that g contains no nontrivial invariant subgroup of the group G, and denote by G/g the homogeneous space for which g is the stabilizer subgroup of one of its points M.

2. Let the index $\hat{\lambda}$ run through a certain (arbitrary) subset of the set of values $0, 1, \dots, q$, and let $\bar{\lambda}$ be the complementary subset. The Pfaffian system of equations in the space G/g

$$\omega^{i_{\hat{\lambda}}} = 0 \tag{2}$$

is invariant with respect to all transformations of the group G. In particular, the system of equations (2), in which $\hat{\lambda} = 0, 1, \dots, p-1$ $(p \leq q)$, is completely integrable and determines in G/g systems of imprimitivity with respect to the subgroup g_p. For example, the maximal integral manifold of this system passing through the point M is the set of points obtained from M under all transformations of the subgroup g_p.

3. For any choice of constants $\xi_{\nu\lambda}$ $(\nu, \lambda = 0, 1, \dots, q;\ \nu > \lambda)$, the system of differential equations

$$d_t\omega^{i_\lambda} = C^{i_\lambda}_{k_\lambda l}\omega^{k_\lambda}\omega^l + \xi_{\nu\lambda}C^{i_\lambda}_{k_\lambda l_\nu}\omega^{k_\lambda}\omega^{l_\nu}, \tag{3}$$

where $d_t\omega = -dt \cdot d(\omega/dt)$, defines in G/g the geodesic lines of a completely determined torsion-free affine connection invariant with respect to the group G. The family of connections obtained in this way depends, generally speaking, on $q(q+1)/2$ parameters (but not fewer than on q). We shall call it the family of C'-connections of the space G/g, determined by the series of subgroups

$\mathfrak{g}^\lambda,\ \mathfrak{g}$, equipped with subspaces $H_\lambda,\ H$. In paper [1] a q-parameter subfamily of this family was studied, called the family of C-connections.

4. The geodesics of every C'-connection that satisfy system (2) at some point satisfy it identically. In other words, system (2) determines in G/g a geodesic field of directions for every C'-connection with geodesics (3). In particular, all systems of imprimitivity by subgroups $g_\lambda,\ \lambda = 1, \dots, q$, are completely geodesic submanifolds with respect to the corresponding family of C'-connections.

Consider the subfamily of the family of C'-connections for which all $\xi_{\nu\mu}$ with $\nu \in \bar{\lambda},\ \mu \in \bar{\lambda}$ (see item 2) are equal to zero. This subfamily is characterized by the fact that all geodesics of its connections satisfying system (2) are trajectories of one-parameter subgroups of the group G.

Let the set of values of the index $\bar{\lambda}$, in turn, be divided into two subsets λ', λ'' according to the criterion $\lambda'' \geq \lambda_0,\ \lambda' < \lambda_0$, where λ_0 is a fixed integer. The equalities $\xi_{\lambda'\lambda''} = 1$ single out a subfamily of C'-connections having the property that the field of directions determined by the system

$$\omega^{i\hat{\lambda}} = 0, \qquad \omega^{i\lambda'} = 0 \tag{4}$$

is parallel along all paths satisfying the system of equations

$$\omega^{i\hat{\lambda}} = 0, \qquad \omega^{i\lambda''} = 0. \tag{5}$$

In other words, any vector of the space G/g satisfying system (4) at the initial point, under parallel displacement along a path satisfying system (5), will satisfy system (4) at every point of the path.

In particular, the family of C'-connections singled out by the conditions $\xi_{\nu\mu} = 1$ for $\nu \geq p,\ \mu < p$, is characterized by the absolute parallelism of the systems of imprimitivity by the subgroup g_p.

The propositions formulated have a number of consequences. Take a geodesic line of a C'-connection satisfying system (2), for which $\xi_{\lambda'\lambda''} = 1$. Consider the natural projection of this line into the homogeneous space G/g_p, where p is greater than all values λ' and not greater than the least of the values λ'' (for example, $p = \lambda_0$). This projection will be a geodesic line of one of the C'-connections of the space G/g_p, specified in it by the series of subgroups $g_1, \dots, g_p$, equipped with subspaces $H_1, \dots, H_p$.

In particular, the natural projection of every geodesic of every C'-connection for which $\xi_{\nu\mu} = 1$ for $\nu \geq \lambda_0,\ \mu < \lambda_0$, into the space G/g_{λ_0}, is a geodesic C'-connection of this space.

5. We shall call a subgroup Li f of the group G normal to the subgroup g in its equipment H (^{3}) if the corresponding subalgebra Li $\hat{f}$ decomposes into the direct sum of its intersections with H and with the subalgebra $\hat{g}$. The following propositions hold.

Let the subgroup f be normal to all subgroups $g_\lambda,\ g$ forming the series (see item 1) in their equipments $H_\lambda,\ H$. Subjecting a point M of the space G/g to all transformations of the subgroup f, we obtain in G/g a submanifold completely geodesic for all C'-connections determined by the series of equipped subgroups $g_\lambda,\ g$.

Let, moreover, f be an invariant subgroup of the group G. Denote by $\bar{g}_\lambda, \bar{g}$ the images of the subgroups $g_\lambda,\ g$ under the natural homomorphism of G onto $G/f = \bar{G}$. Denote by $\bar{H}_\lambda, \bar{H}$ the images of the subspaces $H_\lambda,\ H$ under the corresponding homomorphism of algebras $\hat{G} \to \hat{\bar{G}}$. In our case the subspaces $\bar{H}_\lambda, \bar{H}$ determine equipments of the subgroups $\bar{g}_\lambda, \bar{g}$ of the group $\bar{G}$. Every geodesic C'-connection of the space G/g, under the natural mapping of G/g onto $\bar{G}/\bar{g}$, passes into a geodesic C'-connection of the latter

space, determined by a series of subgroups $\bar{g}_\lambda, \bar{g}$, equipped with subspaces $\bar{H}_\lambda, \bar{H}$.

6. Let, as in Sec. 1 with $q = 0$, the basic forms ω^{i_0}, ω^i of the group G be chosen in such a way that the equations $\omega^{i_0} = 0$ determine the subgroup g, and the equations $\omega^i = 0$ its equipping space H. Denote by $\Phi(g, H)$ the subgroup of the group G generated by all elements belonging to one-parameter subgroups along which the equations $\omega^i = 0$ are satisfied. $\Phi(g, H)$ is an invariant subgroup. The equations defining the corresponding Lie subalgebra can be obtained by setting equal to zero all linear combinations of the forms ω^i whose exterior differentials do not contain the forms ω^{i_0}. We denote the intersection $\Phi(g, H) \cap g$ by $\varphi(g, H)$. The following propositions hold.

Let f_{λ_0} be a subgroup of the group G, normal to the subgroup g_{λ_0} from the series g_λ, g in the equipment H_{λ_0}, and let f be a subgroup of the group g_{λ_0}, normal to the subgroups g_μ, g ($\mu = \lambda_0 + 1, \dots, q$) in the equipments $H_\mu \cap \hat{g}_{\lambda_0}, H \cap \hat{g}_{\lambda_0}$, and containing $\varphi(g_{\lambda_0} \cap f_{\lambda_0}, H_{\lambda_0} \cap \hat{f}_{\lambda_0})$. Denote by P the set of points of the space

G/g obtained from a point M by transformations of the group f. Subjecting all points of the manifold P to transformations of the group $\Phi(g_{\lambda_0} \cap f_{\lambda_0}, H_{\lambda_0} \cap \hat{f}_{\lambda_0})$, we obtain a submanifold of the space G/g, totally geodesic with respect to all C'-connections for which the coefficients $\zeta_{\nu\mu} = 1$ for $\nu \geq \lambda_0,\ \mu < \lambda_0$.

The natural generalization of this proposition is formulated as follows. Let $\lambda_1, \lambda_2, \dots, \lambda_s$ be a certain subset of the set of numbers $0, 1, \dots, q$, arranged in increasing order. Let the subgroup f_{λ_1} of the group G be normal to g_{λ_1} in the equipment H_{λ_1}; let the subgroup f_{λ_2} of the group g_{λ_1} contain $\varphi(f_{\lambda_1} \cap g_{\lambda_1}, \hat{f}_{\lambda_1} \cap H_{\lambda_1})$ and be normal to the subgroups g_μ, $\lambda_1 < \mu \leq \lambda_2$; let the subgroup f_{λ_3} of the group g_{λ_2} contain $\varphi(f_{\lambda_2} \cap g_{\lambda_2}, \hat{f}_{\lambda_2} \cap H_{\lambda_2})$ and be normal to the subgroups g_μ, $\lambda_2 < \mu \leq \lambda_3$, and so on; finally, let the subgroup f of the group g_{λ_s} contain $\varphi(f_{\lambda_s} \cap g_{\lambda_s}, \hat{f}_{\lambda_s} \cap H_{\lambda_s})$ and be normal to the subgroups g_μ, g, $\lambda_s < \mu \leq q$. Subjecting the point M to all transformations of the group f, the resulting manifold to all transformations of the group $\Phi(f_{\lambda_s} \cap g_{\lambda_s}, \hat{f}_{\lambda_s} \cap H_{\lambda_s})$, the newly obtained manifold to all transformations of the group

$$\Phi(f_{\lambda_{s-1}} \cap g_{\lambda_{s-1}}, \hat{f}_{\lambda_{s-1}} \cap H_{\lambda_{s-1}})$$

and so on up to and including the group $\Phi(f_{\lambda_1} \cap g_{\lambda_1}, \hat{f}_{\lambda_1} \cap H_{\lambda_1})$, we obtain in the space G/g a submanifold totally geodesic with respect to all C'-connections for which all coefficients $\zeta_{\nu\mu}$ satisfying the conditions $\nu \geq \lambda_a > \mu$ for at least one of the numbers $\lambda_1, \lambda_2, \dots, \lambda_s$ are equal to one.

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Note: Figure translations are in progress. See original paper for figures.