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Abstract

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ON THE FIRST BOUNDARY-VALUE PROBLEM FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS

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The solvability of boundary-value problem (6) (see below) for the nonstationary Navier–Stokes equations with three spatial variables was established by A. A. Kiselev and O. A. Ladyzhenskaya (1). In their work three generalized formulations of the problem were stated and the corresponding theorems on existence and uniqueness of the solution were proved. The differential properties of the latter are characterized by square summability (over the cylinder domain × time or over sections $t = \text{const}$) of certain derivatives. From these results it is easy to infer that, for sufficiently smooth external forces and initial regimes, problem (6) is also solvable in the classical sense. But the authors of (1) did not set themselves the aim of finding as general as possible conditions under which the classical solvability of (6) holds, and the conditions following from their work are very far from sharp. More general conditions for the classical solvability of (6) were found by P. E. Sobolevskii in (2, 3). However, even in them there are restrictions caused not by the essence of the problem but by shortcomings of the apparatus used.

In the present note a new existence theorem is obtained for the classical solution of problem (6). In addition, the result of (4) on the differential properties of "weak solutions" (5) of this problem is refined.*

We use the apparatus of the theory of nonstationary hydrodynamic potentials in three-dimensional space, developed by one of the authors (7). The general scheme for constructing such a theory was taken by him from Leray, who developed an analogous theory in the two-dimensional case (8).** The central place in this theory, both in the two-dimensional and in the three-dimensional case, is occupied by the construction of "second fundamental solutions," by means of which the solution of problem (1) stated below is expressed. At present we have succeeded in substantially simplifying their explicit form. Below these solutions are given in the newly found form.

Let E_n be n-dimensional space, where $n = 2$ or 3. Further, let $x = (x_1 \dots x_n) \in E_n$; D_{n-1} be the half-space (half-plane) $x_n \geq 0$, $\tilde{x} \equiv x_1$ for $n = 2$, $\tilde{x} \equiv (x_1, x_2)$ for $n = 3$. We also introduce the notation:

$$\Gamma_n(\tilde{x}, x_n, t) = \frac{1}{(4\pi\nu)^{n/2}} \exp\left[-\frac{|\tilde{x}|^2 + x_n^2}{4\nu t}\right];$$

$$M_n(\tilde{x}, x_n) = \begin{cases} -\dfrac{1}{4\pi r}, & \text{where } r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \text{for } n = 3, \\ -\dfrac{1}{2\pi} \log \dfrac{1}{r}, & \text{where } r = \sqrt{x_1^2 + x_2^2}, \quad \text{for } n = 2. \end{cases}$$

* It should be said that this result, while not unexpected, is proved rather laboriously. Therefore it is regrettable that in V. I. Yudovich' s note (^{6}) a close result is stated without any indication of the method of proof.

** Another version of the potential theory was developed by O. V. Guseva.

We pose the following problem for D_{n-1}:

$$\begin{gathered} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \operatorname{grad} p = 0, \qquad \operatorname{div} \mathbf{u} = 0, \qquad -\infty < t < \infty; \\ u_i(\tilde{x}, x_n, t)\big|_{x_n=0} = v_i(\tilde{x}, t) \quad \text{for } i < n; \qquad u_n(\tilde{x}, x_n, t)\big|_{x_n=0} = 0. \end{gathered} \tag{1}$$

The solution of this problem, for arbitrary sufficiently smooth and rapidly decreasing functions $v_i(\tilde{x}, t)$, can be represented in the form

$$u_i(\tilde{x}, x_n, t) = \sum_{j=1}^{n-1} \int_{-\infty}^{t} d\tau \int_{E_{n-1}} G_{ij}^{(n)}(\tilde{x} - y, x_n, t - \tau)\, v_j(y, \tau)\, dy,$$

$$p(\tilde{x}, x_n, t) = \sum_{j=1}^{n-1} \left[\int_{-\infty}^{t} d\tau \int_{E_{n-1}} p_j^{(n)}(\tilde{x} - y, x_n, t - \tau)\, v_j(y, \tau)\, dy + \int_{E_{n-1}} q_j^{(n)}(\tilde{x} - y, x_n)\, v_j(y, t)\, dy\right]. \tag{2}$$

The functions $G_{ij}^{(n)}, p_j^{(n)}, q_j^{(n)}$ are defined for $1 \le i \le n$, $1 \le j < n$ by the formulas

$$G_{ij}^{(n)}(\tilde{x}, x_n, t) = -2\nu \frac{\partial \Gamma_n(\tilde{x}, x_n, t)}{\partial x_n} - 4\nu \left[\operatorname{sign} x_n (1 - \delta_{i3}) + \delta_{i3}\right] \times$$

$$\times \frac{\partial}{\partial x_j} \int_{E_{n-1}} dy \int_0^{x_n} \frac{\partial \Gamma_n(y, \xi, t)}{\partial \xi} \frac{\partial}{\partial x_i} M_n(\tilde{x} - y, x_n - \xi)\, d\xi,$$

$$p_j^{(n)}(\tilde{x}, x_n, t) = -\frac{4\nu}{t} \int_{E_{n-1}} \Gamma_n(y, 0, t) \frac{\partial}{\partial x_j} M_n(\tilde{x} - y, x_n)\, dy,$$

$$q_j^{(n)}(\tilde{x}, x_n, t) = -2\nu \text{ sign}\, x_n \frac{\partial^2}{\partial x_j \partial x_n} M_n(\tilde{x}, x_n).$$

We note that, in fact, the functions $G_{ij}^{(n)}, p_j^{(n)}, q_j^{(n)}$ are defined not only in D_{n-1}, but also in the complementary half-space $D_{n-1}^{(-)}$, and formulas (2) give a solution of problem (1) also in $D_{n-1}^{(-)}$.*
For the second fundamental solution the following estimates hold:

$$\sum_{i,j=1}^{n,n-1} \left| D_x^l D_t^m G_{ij}^{(n)}(\tilde{x}, x_n, t) \right| \le \frac{\nu^m C |x_n|^{2\delta - 1}}{(\nu t)^{m+\delta} \left(r^2 + \nu t\right)^{\frac{n+l}{2}}},$$

$$\sum_{i,j=1}^{n,n-1} \left| D_x^l D_{x_n}^{l_n} D_t^m G_{ij}^{(n)}(\tilde{x}, x_n, t) \right| \le \frac{C\nu^m}{(\nu t)^{m+\frac{l_n+1}{2}} \left(r^2 + \nu t\right)^{\frac{n+1}{2}}}.$$

Here $\delta \in [1/2, 1]$; $r^2 = \sum_{i=1}^n x_i^2$; D_x^l is any derivative of order l with respect to the corresponding variables.

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* In Leray' s work these functions were found only in D_1, as a result of which, in what follows, the domain in which the solution is sought is assumed to be convex. This restriction can now be removed.

Let Ω be a bounded domain in E_3, bounded by a Lyapunov surface S with exponent α. Consider the problem

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \operatorname{grad} p = \mathbf{f}(x, t), \qquad \operatorname{div} \mathbf{u}(x) = 0,$$

$$\mathbf{u}\big|_S = \mathbf{u}(s, t), \qquad \mathbf{u}\big|_{t=0} = \mathbf{a}(x), \qquad x \in \Omega, \quad t > 0, \tag{3}$$

and suppose that the conditions

$$\int_S (\mathbf{u}(s, t) \cdot \mathbf{n}(s))\, ds \equiv 0, \qquad \operatorname{div} \mathbf{a}(x) \equiv 0$$

are satisfied, where $\mathbf{n}(s)$ is the unit normal to S (the second equality must hold in the generalized sense of S. L. Sobolev).

In (^{7}) problem (3) was considered for $\mathbf{f} \equiv 0$ and $\mathbf{a} \equiv 0$, and it was shown that its solution $\mathbf{u}$ can be represented as the sum of the gradient of a harmonic function and "potentials," whose kernels are expressed in terms of $G_j^{(3)}$. The "densities" are found from a system of integral equations of Volterra–Fredholm type, which can be solved by the method of successive approximations.

Here we give estimates for the solution of problem (3) for $\mathbf{f} \not\equiv 0$ and $\mathbf{a} \not\equiv 0$. These cases are easily reduced to the preceding one, as was done in (^{8,9}).

Introduce the following notation: $M(\Omega, \beta)$ is the space of vector-functions $\mathbf{v}(x)$, defined in $\Omega + S$, with finite norm

$$\max_{x,x' \in \Omega+S} \frac{|\mathbf{v}(x) - \mathbf{v}(x')|}{|x - x'|^{\beta}} + \max_{x \in \Omega+S} |\mathbf{v}(x)| = \|\mathbf{v}(x)\|_{M(\Omega,\beta)},$$

$$\|\mathbf{u}(x,t)\|_{M(\Omega,\beta)} \equiv V_{\beta}(t).$$

1. Let $\mathbf{u}(x,t)\big|_S \equiv 0$, $\mathbf{f}(x,t) \equiv 0$, $\mathbf{a}(x) \in M(\Omega,\beta)$. Then

$$V_{\beta'}(t) \leq B\|\mathbf{a}(x)\|_{M(\Omega,\beta)} e^{-\gamma\nu t}, \tag{4}$$

where B is a constant depending on Ω, β, and β', while γ is another constant depending only on Ω.

2. Let $\mathbf{u}(x,t)\big|_S \equiv 0$, $\mathbf{f}(x,t) \equiv 0$, $\mathbf{a}(x) \in L_p(\Omega)$. Then, if $\beta + 3/2p < 1$ and $\beta' < \beta \leq \alpha$, we have

$$V_{\beta'}(t) \leq B\left(1 + \frac{1}{(\nu t)^{\beta+3/2p}}\right)\|\mathbf{a}(x)\|_{L_p} e^{-\gamma\nu t}. \tag{4'}$$

3. Let $\mathbf{u}(x,t)\big|_S \equiv 0$, $\mathbf{a}(x) \equiv 0$,

$$f_i(x,t) = \frac{\partial R_{i,j}}{\partial x_j} + F_i,$$

where

$$\sum_{i=1}^{3} \|R_i\|_{M(\Omega,\beta)} \leq \varphi(t), \qquad \|\mathbf{F}\|_{M(\Omega,\beta)} \leq \psi(t),$$

where $R_i \equiv (R_{i,1}, R_{i,2}, R_{i,3})$. Then

$$V_{\beta}(t) \leq \int_0^t \left\{\frac{B_{\delta}\,\varepsilon[\nu(t-\tau)]}{[\nu(t-\tau)]^{1/2+\delta}} + Be^{-\gamma\nu(t-\tau)}\right\}\varphi(\tau)\,d\tau +$$

$$+\int_0^t \left\{\frac{B_{\delta}\,\varepsilon[\nu(t-\tau)]}{[\nu(t-\tau)]^{\delta}} + Be^{-\gamma\nu(t-\tau)}\right\}\psi(\tau)\,d\tau, \tag{5}$$

where $\varepsilon(y) = 1$ for $y \leq 1$; $\varepsilon(y) = 0$ for $y > 1$; and δ is an arbitrarily small positive number.

Results (4) and (5) make it possible to prove the classical solvability of the following problem:

$$\frac{\partial u}{\partial t} - \nu \Delta u + \operatorname{grad} p = u_k \frac{\partial u}{\partial x_k} + f, \qquad \operatorname{div} u = 0,$$

$$u|_S = 0, \qquad u|_{t=0} = a \quad (\operatorname{div} a(x) \equiv 0). \tag{6}$$

For this it is sufficient to consider the following sequence of linear problems, corresponding to $n = 0, 1, 2, \ldots$:

$$\frac{\partial u_{n+1}}{\partial t} - \nu \Delta u_{n+1} + \operatorname{grad} p_{n+1} = u_{n,k} \frac{\partial u_n}{\partial x_k} - f,$$

$$\operatorname{div} u_{n+1} = 0, \qquad u_{n+1}|_S = 0, \qquad u_{n+1}|_{t=1} = a, \tag{7}$$

putting $u_0(x,t) \equiv 0$.

The convergence of this process is investigated in the same way as in (^{8}), and is connected with the use of estimates (4) and (5).

Theorem. Suppose

$$\sup_{t>0} \|f(x,t)\|_{M(\Omega;\beta)} < \infty, \qquad \sup_{t,t'>0} \max_{x \in \bar{\Omega}} \frac{|f(x,t') - f(x,t)|}{|t-t'|^{\beta}} < \infty,$$

$$\|a(x)\|_{M(\Omega,\beta)} < \infty$$

for some $\beta > 0$. Then in some cylinder $Q \equiv (\Omega \times [0,T])$ there exists a classical (i.e. continuous up to the boundary S and the plane $t = 0$, and having in Ω the continuous derivatives entering into the equations) solution of problem (6). T is estimated from below in terms of the quantities $\sup_{t>0} \|f(x,t)\|_{M(\Omega,\beta)}$ and $\|a(x)\|_{M(\Omega,\beta)}$, and in the case of sufficient smallness of the latter $T = \infty$.

Using estimate (4′), one can prove that process (7) converges to a solution of problem (6) if $a(x) \in L_p(\Omega)$ for $p > 3$. In this case the solution satisfies the initial condition in the weak sense, and the equations and boundary condition according to the nature of f.

Let us now consider problem (3), assuming that $(u(s,t) \cdot n(s)) \equiv 0$, and that S is three times continuously differentiable.

Then, using the methods developed in (^{10}), one can prove the following estimate for $1 < p \neq 3/2$:

$$\sum_{i,j=1}^{3} \|u_{x_i x_j}\|_{L_p(Q)} + \|u_t\|_{L_p(Q)} + \|\operatorname{grad} p\|_{L_p(Q)} \leq$$

$$\leq C\left[\|f\|_{L_p(Q)} + \|a\|_{W_p^{2-2/p}(\Omega)} + \|u(s,t)\|_{W_p^{2-1/p,\,1-1/2p}(S\times[0,T])}\right].$$

Hence, as in (^{4}), it follows that

Theorem. The "weak" solution of problem (6) has derivatives $u_{x_i x_j}$, u_t, p_{x_i}, summable with power $5/4$ over Q.

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