MATHEMATICS

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Abstract

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THEOREMS ON THE SEPARABILITY OF CONVEX POLYHEDRAL SETS

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1. Let L be an arbitrary real linear (vector) space and

$$f_j(x) - a_j \leqslant 0 \qquad (j = 1, 2, \dots, m) \tag{1}$$

an arbitrary system of linear inequalities over L, i.e., a system in which $f_1(x), \dots, f_m(x)$ are real linear functions (linear functionals) defined on L, and $a_1, a_2, \dots, a_m$ are real numbers. If the rank r of system (1), i.e., the maximal number of linearly independent functions among $f_1(x), \dots, f_m(x)$, is nonzero, then the set M of its solutions in L will be called a **convex polyhedral set** of the space L; for $m = 1$ such a set will be called a half-space of the space L.

An arbitrary plane of the space L, i.e., the set of all elements of L satisfying an arbitrary equation $f(x) - a = 0$, where $f(x)$ is a nonzero linear functional defined on L, and a is a real number, divides the space L into two half-spaces, defined respectively by the inequalities $f(x) - a \leqslant 0$ and $-f(x) + a \leqslant 0$; the interior elements of these half-spaces are defined by the inequalities $f(x) - a < 0$ and $-f(x) + a < 0$.

The set M of solutions of system (1) will be called a **convex polyhedral cone** if the system of its boundary equations $f_j(x) - a_j = 0$ is consistent. We shall say of a cone determined by some subsystem of system (1) that it embraces the set M of solutions of system (1). If the number of inequalities and the rank of the subsystem defining such a cone coincide with the rank of system (1), and all solutions of the system of its boundary equations satisfy system (1), then we shall call it a **corner cone** of the set M.

Theorem 1. *If two convex polyhedral sets of an arbitrary real linear space L have no common elements, then there exists a plane of the space L that strictly separates these sets: one of them is contained inside one of the two half-spaces determined by this plane in L, and the other—inside the other.*

This is proved with the aid of Theorem 2 from paper ([1]).

Theorem 2. *If two convex polyhedral sets of an arbitrary real linear space L have no common elements, then among the cones embracing these sets one can* *distinguish, respectively, at least two corner cones having no common elements.*

This proposition follows from Theorem 1 by virtue of Corollary 1 of Theorem 1 of paper ([1]).

2. Two convex sets A and B of a real linear space L having common points will be called **tangent** in L if there exists a vector $a \in L$ such that, when one of them is shifted by the vector at with any $t > 0$, a set is obtained that has no common ⋯

elements with another. It should be noted that the same result is obtained when the latter is shifted by the vector $-at$.

Theorem 3. *If two convex polyhedral sets of an arbitrary real linear space L are tangent, then among their containing cones one can select, respectively, at least two pointed cones tangent to one another.*

Theorem 4. *For any two tangent convex polyhedral sets A and B in L, there exists a plane such that one of them is contained in one of the two half-spaces determined by it in L, and the other in the other.*

3. Two systems of linear inequalities

$$f_j(x) - a_j \leq 0 \qquad (j = 1, 2, \dots, m); \tag{2}$$

$$F_i(x) - b_i \leq 0 \qquad (i = 1, 2, \dots, n) \tag{3}$$

over L, having respectively ranks r and s, will be called **mutually intertwined** if, to an arbitrary identically zero linear combination

$$p_{j_1} f_{j_1}(x) + \dots + p_{j_r} f_{j_r}(x) + q_{i_1} F_{i_1}(x) + \dots + q_{i_s} F_{i_s}(x)$$

with nonnegative coefficients, at least one of which is different from zero (a positive linear combination), containing r linearly independent functions $f_j(x)$ and s linearly independent functions $F_i(x)$, there corresponds the relation

$$p_{j_1} a_{j_1} + \dots + p_{j_r} a_{j_r} + q_{i_1} b_{i_1} + \dots + q_{i_s} b_{i_s} \geq 0,$$

or if from the functions entering the systems (2) and (3) one cannot form any identically zero combination of this kind.

Theorem 5. *If each of two mutually intertwined systems (2) and (3) is consistent, then the system*

$$\begin{aligned} f_j(x) - a_j \leq 0 \qquad & (j = 1, 2, \dots, m); \\ F_i(x) - b_i \leq 0 \qquad & (i = 1, 2, \dots, n), \end{aligned}$$

obtained by joining them, is also consistent. Conversely, if this system is consistent, then the systems (2) and (3) are mutually intertwined.

This is proved with the aid of Theorem 2.

4. A system of linear inequalities will be called **proper** if it contains more than one inequality and does not contain inequalities $f(x) - a \leq 0$ with the zero linear function $f(x)$ and a number $a < 0$ (a contradictory inequality).

Theorem 6. *Every proper system of linear inequalities*

$$f_j(x) - a_j \leq 0 \qquad (j = 1, 2, \dots, m) \tag{4}$$

(over L) is the union of two consistent subsystems having no inequalities in common.

In the case of Euclidean space $L = R^n$, i.e. for the case of a system (4) of the form

$$f_j(x) - a_j \equiv a_{j1}x_1 + \dots + a_{jn}x_n - a_j \leq 0 \qquad (j = 1, 2, \dots, m), \tag{5}$$

two such subsystems are selected in the following way.

We shall agree to call an elementary transformation of the system (5) the reduction of it to the form

$$a_{j1}x_1 + \dots + (a_{jk} + a \cdot a_{jl})x_k + \dots + a_{jl}u_l + \dots + a_{jn}x_n - a_j \leq 0 \tag{6}$$

$$(j = 1, 2, \dots, m),$$

where a is an arbitrary real number and $u_l = x_l - ax_k$. It is clear that the system (5) is consistent if and only if the system (6) is consistent.

It is not difficult to see that every system (5) containing more than one inequality and containing no inequalities with zero linear form can, by means of a sequence of suitably chosen elementary transformations, be transformed into a system in which the coefficients of at least one unknown are nonzero. If all these coefficients have the same sign, then the new system is consistent. But then the original system (5) is also consistent. In this case, any partition of system (5) into two subsystems without common inequalities answers the question that interests us.

If not all the coefficients under consideration have one and the same sign, then, by selecting in the new system two subsystems in which these coefficients have the same sign, and passing to the corresponding subsystems of system (5), we obtain the desired partition of the latter.

If the regular system (5) contains noncontradictory inequalities with zero linear form, then the preceding arguments can be carried out for its maximal subsystem not containing such inequalities (which exists if the latter contains more than one inequality), and then all inequalities with zero linear form can be adjoined to one of the two subsystems into which this subsystem is divided. If the distinguished maximal subsystem contains only one inequality, then we take it as one of the subsystems of the desired partition of system (5).

Let now L be an arbitrary real linear space, R its maximal subspace on which all the functions $f_j(x)$ entering the regular system (4) vanish, and S some direct complement in L of this subspace. The dimension of the subspace S obviously coincides with the rank r of system (4). If $r = 0$, then any partition of system (4) into two subsystems having no common inequalities satisfies the condition of Theorem 6.

If $r > 0$, then choose in S some basis $x_1, \dots, x_r$. Then the system

$$t_1 f_j(x_1) + \dots + t_r f_j(x_r) - a_j \leq 0 \qquad (j = 1, 2, \dots, m), \tag{7}$$

where $t_1, \dots, t_r$ are unknowns taking real values, is consistent if system (4) is consistent. If $(t_1, \dots, t_r)$ is an arbitrary solution of system (7), then $t_1 x_1 + \dots + t_{rx} r + u$ $(u \in R)$ is an arbitrary solution of system (4).

System (4) is regular if and only if system (7) is regular. If system (7) is divided in the manner described above into two consistent subsystems having no common inequalities, then the two corresponding subsystems of system (4) must satisfy the conditions of Theorem 6 (this follows from the relation between systems (4) and (7) noted here).

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CITED LITERATURE

1. S. N. Chernikov, DAN, 131, No. 3, 518 (1960).

Note: Figure translations are in progress. See original paper for figures.