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Abstract

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KLEIN SPACES AS KAWAGUCHI SPACES

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It is known that in every Klein space K_n with a finite group of automorphisms G one can construct n or $n+1$ functions of the form $L(\xi^\alpha, \xi^{(i)\alpha})$ $(\alpha, \beta = 1, 2, \ldots, n;\ i = 1, 2, \ldots, r)$, possessing the property that the integral over any oriented curve $\xi^\alpha = \xi^\alpha(t)$ $(t_1 \leq t \leq t_2)$

$$\int_{t_1}^{t_2} L\left(\xi^\alpha, \frac{d^i \xi^\alpha(t)}{dt^i}\right) dt \tag{1}$$

does not depend on the choice of the parameter t of the curve and is unchanged if the curve is transformed by automorphisms of K_n; moreover, any other function possessing these properties is expressible in terms of the chosen functions and their differential consequences. Each of such functions may be used to obtain an invariant parameter–the "arc length" of the curve–and the ratios of these functions give $n-1$ or n functionally independent invariants–the "curvatures" of the curves.

Since the specification of the integral (1) determines the structure of a Kawaguchi space with metric $L(\xi^\alpha, \xi^{i(\alpha)})$, it follows that in every K_n one can construct such a complete system of n or $n+1$ Kawaguchi metrics that any Kawaguchi metric of the given K_n is expressed in terms of these metrics. If L takes only nonnegative values, then specifying the Kawaguchi metric L is equivalent to specifying a field of hypersurfaces–the indicatrices of this metric in the associated manifold $T_{\nu n}(X_n)$ ([1]), defined by the equation $L(\xi^\alpha, \xi^{(i)\alpha}) = 1$.

Let us show that the group $\bar{G}$ of all transformations of K_n with respect to which all Kawaguchi metrics of some complete system are invariant coincides with G. Indeed, if one takes a curve that is not a trajectory of some one-parameter subgroup of $\bar{G}$, then the set of all its images under transformations of $\bar{G}$ has the same curvatures as functions of the chosen arc length, and therefore is obtained from the original curve by means of transformations of G, which is possible only when $\bar{G}$ coincides with G.

We shall call a transformation of a Kawaguchi space **conformal** if under it the metric changes in the following way:

$$\bar{L} = \sigma(\xi^\alpha) L.$$

A conformal transformation of a Kawaguchi space is called **nontrivial** if it is not an automorphism of this space, i.e. if for it $\sigma(\xi^\alpha) \neq 1$.

Theorem. *In every Klein space K_n with automorphism group G one can construct a Kawaguchi metric that has no nontrivial conformal transformations and whose automorphism group coincides with G.*

We shall first prove the theorem in the case when G is transitive. We shall assume that the complete system of Kawaguchi metrics of the given K_n is such that the group of transformations under which all of them transform conformally with the same-

···by the same factors $\sigma(\xi^\alpha)$, coincides with G, since otherwise we can achieve this by adding to the system the metric

$$\frac{d}{dt} K = \frac{\partial K}{\partial \xi^{(1)\alpha}} \xi^{(1)\alpha} + \frac{\partial K}{\partial \xi^{(1)\alpha}} \xi^{(2)\alpha} + \cdots + \frac{\partial K}{\partial \xi^{(\nu)\alpha}} \xi^{(\nu+1)\alpha},$$

where K is one of the curvatures of our system. Clearly, it now remains only to show that from two Kawaguchi metrics L_1 and L_2 one can always construct a metric whose group of conformal transformations coincides with the group of transformations of K_n under which L_1 and L_2 are transformed conformally with the same factors $\sigma(\xi^\alpha)$.

Consider the metric $L = f(K) L_1$, where $K = L_1 / L_2$ and f is some differentiable function. Let the vector field v^α define a one-parameter group of conformal transformations of the metric L. Then, denoting by $\overset{L}{D}$ the symbol of the Lie derivative with respect to the vector field corresponding to the differential prolongation of the given group of the proper order, we obtain

$$\overset{L}{D} L = f'(K) \overset{L}{D} K L_1 + f(K) \overset{L}{D} L_1 = \lambda(\xi^\alpha) L,$$

where $\lambda(\xi^\alpha)$ is some function determined by the group. If, under the transformations of the given group, L_1 and L_2 are not transformed conformally with the same factors $\sigma(\xi^\alpha)$, then $\overset{L}{D} K \neq 0$ and the equality holds

$$\frac{d}{dK} \ln f(K) = \frac{\lambda(\xi^\alpha)}{\overset{L}{D} K} - \frac{\overset{L}{D} L_1}{L_1 \overset{L}{D} K}. \tag{2}$$

If this equality is considered at some point ξ_0^α, then its right-hand side, for all possible λ and v^α, determines a family of functions of the variables $\xi^{(i)\alpha}$, depending on a finite number of parameters. The left-hand side of the equality is an arbitrary differentiable function of K, and K, in view of the transitivity of the group G, essentially depends on $\xi^{(i)\alpha}$. Therefore one can always, and moreover with functional arbitrariness, choose $f(K)$ so that (2) cannot be satisfied for any v^α, and consequently L has the required properties.

If the group G is intransitive, then one may assume that at least two Kawaguchi metrics of the complete system are such that their ratio essentially depends on $\xi^{(i)\alpha}$, since otherwise we would obtain such a system by adding a new metric whose order is higher than the order of all metrics of the complete system. Constructing, for these two metrics, a metric L in the indicated way, one can choose $f(K)$ so that the ratio of L and at least one of the remaining metrics of the system essentially depends on $\xi^{(i)\alpha}$, and then the process of constructing the desired Kawaguchi metric can be continued. Thus the theorem is proved.

We shall call a Kawaguchi metric L a **characteristic metric of the Klein space** K_n if the group of its automorphisms coincides with G.

Let ξ^α be some point of K_n. The group of automorphisms of the tangent $T_{\nu n}$ at the point ξ^α, carrying the indicatrix of some characteristic Kawaguchi metric L of the given K_n at this point into itself, determines, up to similarity[(1)], a local geometric object in $T_{\nu n}$. The images of this object under transformations of G determine a homogeneous field of such an object in the transitivity class of the group G of the point ξ^α. Construct the field of such objects in all transitivity classes of the group G, i.e. the field of some differential-geometric object on all of K_n. The homogeneity sets of this field coincide with the transitivity classes of G, and the invariance group of the field with the group G. For any field of a differential-geometrical···

of an object locally equivalent to the given field, one can construct the field of indicatrices of a certain Kawaguchi metric, invariant with respect to those and only those automorphisms $T_{\nu n}$ which belong to the invariance group of the corresponding local object of the field; moreover, the indicatrices of this field differ from the corresponding indicatrices of the original characteristic Kawaguchi metric L only by automorphisms $T_{\nu n}$. The Kawaguchi metric constructed in this way determines the original field of a differential-geometric object up to homogeneous-local similarity.

It follows from this that for every X_n with a given field of a differential-geometric object Ω that admits a mapping onto some K_n under which the image of Ω is locally equivalent to the field of a characteristic object of K_n, one can construct a characteristic Kawaguchi metric that determines, up to homogeneous-local similarity, the field of the object Ω.

In paper ([2]) examples are given of characteristic Kawaguchi metrics for equiaffine and symplectic spaces and for the corresponding spaces with curvature. It can be shown that, as a characteristic metric of a space of symmetric affine connection A_n, one may take the following Kawaguchi metric:

$$L = \left| \frac{2n(n^2-1)\mathfrak{A}_{12\ldots n-1\,n+2}\mathfrak{A}_{12\ldots n} + 2n(n+1)(n+2)\mathfrak{A}_{12\ldots n-2\,n\,n+1}\mathfrak{A}_{12\ldots n}}{\mathfrak{A}^2_{12\ldots n}} - \frac{(n+2)(2n^2-n-1)\mathfrak{A}^2_{12\ldots n-1\,n+1}}{\mathfrak{A}^2_{12\ldots n}} \right|^{1} \quad (3)$$

where

$$\mathfrak{A}_{i_1 i_2 \ldots i_n} = \mathrm{Det} \left| \frac{\delta^{i_1}\xi^\alpha}{dt^{i_1}}, \frac{\delta^{i_2}\xi^\alpha}{dt^{i_2}}, \ldots, \frac{\delta^{i_n}\xi^\alpha}{dt^{i_n}} \right|,$$

and $\frac{\delta}{dt}$ is the symbol of absolute differentiation with respect to the given affine connection. If A_n has zero curvature, (3) determines a characteristic Kawaguchi metric of an affine space. As a characteristic Kawaguchi metric of A_n with a given field of a contravariant vector v^α, one may take the following Kawaguchi metric:

$$L = \left| \frac{\mathfrak{A}_{12\ldots n}}{\mathfrak{A}} \right|^{1/n}, \quad (4)$$

where

$$\mathfrak{A} = \mathrm{Det} \left| v^\alpha, \frac{\delta\xi^\alpha}{dt}, \ldots, \frac{\delta^{\,n-1}\xi^\alpha}{dt^{\,n-1}} \right|.$$

If the connection in this space has zero curvature and $\nabla_\beta v^\alpha = \delta^\alpha_\beta$, then (4) determines a characteristic Kawaguchi metric of a centro-affine space.

It should be noted that in the examples indicated the Kawaguchi metrics have the least differential order among all Kawaguchi metrics determined by the given spaces, which, generally speaking, is not always possible.

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CITED LITERATURE

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2. M. V. Losik, DAN, **134**, No. 6, 1299 (1960).

Note: Figure translations are in progress. See original paper for figures.