A CHARACTERIZATION OF FINITELY AXIOMATIZABLE CLASSES OF MODELS

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1961

Abstract

Full Text

MATHEMATICS

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A CHARACTERIZATION OF FINITELY AXIOMATIZABLE CLASSES OF MODELS

(Presented by Academician A. I. Mal' cev, 15 XII 1960)

In the note ([1]) a characterization was given of finitely axiomatizable classes of models for certain kinds of axioms. In the present note a characterization is given of finitely axiomatizable classes of models without restriction on the kinds of axioms.

A finite system of axioms of the narrow predicate calculus is equivalent to a single axiom of the form

$$\forall x_1^1 x_2^1 \dots x_{n_1}^1 E x_1^2 \dots x_{n_2}^2 \dots Q x_1^l \dots x_{n_l}^l \ \nu(x_1^1, \dots, x_{n_1}^1, \dots, x_{n_l}^l) \tag{1}$$

or of the form

$$E x_1^1 x_2^1 \dots x_{n_1}^1 \ \forall x_1^2 \dots x_{n_2}^2 \dots Q x_1^l \dots x_{n_l}^l \ \nu(x_1^1, \dots, x_{n_l}^1, \dots, x_{n_l}^l), \tag{2}$$

where $Q = \forall$ or E, depending on the parity of l. Therefore it is enough to characterize the class of models described by an axiom of the form (1) or (2).

1. To characterize the class of models described by an axiom of the form (1), we introduce the following definition, which is a strengthening of the analogous definition from ([1]).

Definition 1. For a given tuple $(n_1, n_2, \dots, n_l)$, a given model $\mathfrak{M}$, and a submodel $\mathfrak{N}_{n_1}$, $\mathfrak{N}_{n_1} \in S_{n_1}(\mathfrak{M})$, we define the notion of an $(n_1, n_2, \dots, n_{l-1}, n_l)$-covering of the model $\mathfrak{M}$ with respect to the submodel $\mathfrak{N}_{n_1} = (a_1, a_2, \dots, a_{n_1})$. For $l = 2$ the model $\mathfrak{M}$ is called an (n_1, n_2)-covering of the model $\mathfrak{R}$ with respect to $\mathfrak{N}_{n_1}$, if there exists an isomorphic mapping φ_1 of the model $\mathfrak{N}_{n_1}$ into $\mathfrak{M}$ such that for every n_2-extension $\mathfrak{M}_{n_1 n_2}$ (containing not more than $n_1 + n_2$ elements) of the model $\varphi(\mathfrak{N}_{n_1})$ in $\mathfrak{M}$ there exists an isomorphic mapping φ_2 of the model $\mathfrak{M}_{n_1 n_2}$ into $\mathfrak{R}$, coinciding with φ_1^{-1} on $\mathfrak{N}_{n_1}$. We shall write this briefly as follows:

$$\mathfrak{R} \preccurlyeq_{(\mathfrak{N}_{n_1}, n_1, n_2)} \mathfrak{M}.$$

The mappings φ_1, φ_2 will be called admissible. Suppose that an $(n_1, n_2, \dots, n_{l-1})$-covering has been defined, and define an $(n_1, n_2, \dots, n_l)$-covering.

Consider two cases:

1) $l - 1 = 2s$. Then $\mathfrak{R} \preccurlyeq_{(\mathfrak{N}_{n_1}, n_1, n_2, \dots, n_l)} \mathfrak{M}$ means that $\mathfrak{R} \preccurlyeq_{\mathfrak{N}_{n_1}, n_1, \dots, n_{l-1}} \mathfrak{M}$, and for every sequence of admissible mappings $\varphi_1, \varphi_2, \dots, \varphi_{l-1}$, for every n_l-extension $\mathfrak{M}_{n_1 n_2 \dots n_{l-1} n_l}$ of the model $\mathfrak{N}_{n_1 n_2 \dots n_{l-1}}$ in $\mathfrak{M}$, there exists an isomorphic mapping φ_l of the model $\mathfrak{N}_{n_1 n_2 \dots n_{l-1}}$ into $\mathfrak{M}$, coinciding with φ_{l-1}^{-1} on the model $\mathfrak{N}_{n_1 n_2 \dots n_{l-1}}$.
2) $l - 1 = 2s + 1$. Then $\mathfrak{R} \preccurlyeq_{(\mathfrak{N}_{n_1}, n_1, n_2, \dots, n_l)} \mathfrak{M}$ means that $\mathfrak{R} \preccurlyeq_{(\mathfrak{N}_{n_1}, n_1, \dots, n_{l-1})} \mathfrak{M}$, and for every sequence of admissible mappings $\varphi_1, \varphi_2, \dots, \varphi_{l-1}$, for every n_l-extension $\mathfrak{M}_{n_1 n_2 \dots n_{l-1} n_l}$ of the model $\mathfrak{N}_{n_1 n_2 \dots n_{l-1}}$ in $\mathfrak{M}$, there exists an isomorphic mapping φ_l of the model $\mathfrak{M}_{n_1 n_2 \dots n_{l-1} n_l}$ into $\mathfrak{R}$, coinciding with φ_{l-1}^{-1} on the model $\mathfrak{N}_{n_1 n_2 \dots n_{l-1}}$. Thus, for every tuple $(n_1, n_2, \dots, n_l)$

defined an $(n_1, n_2, \dots, n_l)$-covering of the model $\mathfrak{M}$ with respect to $\mathfrak{M}_{n_1}$, i.e.

$$\mathfrak{N} \leqslant_{(\mathfrak{M}_{n_1}, n_1, n_2, \dots, n_l)} \mathfrak{M}. \tag{3}$$

Definition 2. If, for the given tuple $(n_1, n_2, \dots, n_l)$ and for any submodel $\mathfrak{M}_{n_1}$ from $S_{n_1}(\mathfrak{M})$, (1) holds, then we write

$$\mathfrak{N} \leqslant_{(n_1, n_2, \dots, n_l)} \mathfrak{M}. \tag{4}$$

If, for any tuple $(n_1, n_2, \dots, n_l)$ of fixed rank l, inequality (4) holds, then $\mathfrak{N} \leqslant_{(l)} \mathfrak{M}$.

Let L be the class of all models of type $\mathfrak{M} = \langle M, P \rangle$. For a given number n there exists in the class L a finite number $\varphi_L(n)$ of pairwise nonisomorphic models containing no more than n elements. For given models $\mathfrak{M}$ from L and a submodel $\mathfrak{M}_n$ from $S_n(\mathfrak{M})$, there exist no more than $\varphi_L(n + m)$ pairwise nonisomorphic m-extensions of the model $\mathfrak{M}_n$ in the model $\mathfrak{M}$.

Lemma 1. *The class of all models $\mathfrak{M}$ satisfying the condition $\mathfrak{N} \leqslant_{(\mathfrak{M}_{n_1}, n_1, \dots, n_l)} \mathfrak{M}$ for given $\mathfrak{N}, \mathfrak{M}_{n_1}, n_1, n_2, \dots, n_l$ is described by an axiom of the form*

$$Ex_1^1 \dots x_{n_1}^1 \, \forall x_1^2 \dots x_{n_2}^2 \, Ex_1^3 \dots x_{m_3}^3 \, \forall x_1^4 \dots x_{n_4}^4 \, Ex_1^5 \dots x_{m_5}^5 \, \dots$$

$$\dots Qx_1^l \dots x_{m_l}^l \, v(x_1^1, \dots, x_{n_l}^l),$$

where $Q = E$ if l is odd, and $Q = \forall$ if l is even; $v(x_1^1, x_2^1, \dots, x_{n_l}^l)$ contains no quantifiers; the number m_i is not greater than

$$n_i \varphi_L(n_1+n_2)\,\varphi_L(n_1+n_2+n_3)\cdots\varphi_L(n_1+n_2+\cdots+n_i), \quad i=3,5,7,\dots.$$

This lemma is a strengthening of the analogous lemma from ([1]). Hence it follows:

Theorem 1. *The following conditions are equivalent:*

1) *The class K is described by an axiom of the form (1).*
2) *There exists a number N such that, if for a model $\mathfrak{A}$ and any of its submodels $\mathfrak{A}'$ from $S_N(\mathfrak{A})$ (containing no more than N elements) there is found in K a model $\mathfrak{B}_{\mathfrak{A}'}$ satisfying the condition*

$$\mathfrak{A} \leqslant_{(\mathfrak{A}',\underbrace{N,N,\dots,N}_{l})} \mathfrak{B}_{\mathfrak{A}'},$$

then $\mathfrak{A} \in K$.

2. An axiom of the form (2) may be regarded as an axiom of the form (1), where the number of variables x^1 is equal to zero. Therefore the following holds:

Theorem 2. *The following conditions are equivalent:*

1) *The class K is described by an axiom of the form (2).*
2) *There exists a number N such that, if for a model $\mathfrak{A}$ one can indicate a model $\mathfrak{B}_{\mathfrak{A}}$ from K satisfying the condition*

$$\mathfrak{B}_{\mathfrak{A}} \leqslant_{(\underbrace{N,N,\dots,N}_{l})} \mathfrak{A}, \tag{5}$$

then $\mathfrak{A} \in K$.

From these theorems follow the characterizations of finitely axiomatizable classes of models given in ([1,2]).

3. In order to obtain a strengthening of the Łoś–Tarski theorem, we introduce the following definitions:

Definition 3. A submodel $\mathfrak{N}$ of a model $\mathfrak{M}$ is called an (N,l)-submodel if $\mathfrak{N} \leqslant_{(\underbrace{N,N,\dots,N}_{l})} \mathfrak{M}$. The class of all (N,l)-submodels of the model $\mathfrak{M}$ will be denoted by $S_{(N,l)}(\mathfrak{M})$. Then

$$\mathfrak{N} \in S_{(N,l)}(K) \equiv (E\mathfrak{M})(\mathfrak{M} \in K \;\&\; \mathfrak{N} \in S_{(N,l)}(\mathfrak{M})).$$

Definition 4. A class of models K is called an (N,l)-class if it has the following property: if, for a model $\mathfrak{A}$, any submodel $\mathfrak{A}_N$

from $S_N(\mathfrak{A})$ in the class K there is a model
$\mathfrak{M}_{\mathfrak{A},\mathfrak{A}_1,\underbrace{N,N,\dots,N}_{l}}$ such that

$$\mathfrak{A} \leqslant_{\langle \mathfrak{A}_N, \underbrace{N,N,\dots,N}_{l} \rangle} \mathfrak{M}_{\mathfrak{A},\mathfrak{A}_N,\underbrace{N,\dots,N}_{l}},$$

then there exists a model $\mathfrak{M}$ in K containing $\mathfrak{A}$ and

$$\mathfrak{A} \leqslant \langle \underbrace{N, N, \dots, N}_{l} \rangle \mathfrak{M},$$

i.e. $\mathfrak{A} \in S_{(N,l)}(\mathfrak{M})$.

Theorem 3. The following conditions are equivalent:
1) The class K is defined by an axiom of the form (1).
2) There exist numbers N, l such that K is an (N, l)-class and $S_{N,l}(K) \subset K$.

Proof. If the class K is described by an axiom of the form (1), then K is an (N, l)-class for some pair of numbers N, l.

This assertion is an analogue of Henkin' s theorem.

Theorem 4. If K is an (N, l)-class for some pair of numbers N, l, then $S_{(N,l)}(K)$ is described by an axiom of the form (1).

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Received
10 XII 1960

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Note: Figure translations are in progress. See original paper for figures.