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Abstract

Full Text

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DIVERGENT FOURIER SERIES OF CONTINUOUS FUNCTIONS

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In 1927 A. N. Kolmogorov, in a paper [1] joint with D. E. Menshov, formulated the following assertion:

A. N. Kolmogorov' s Theorem. There exists a function $f(x) \in L^2[0, 2\pi]$ whose Fourier series

$$\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{1}$$

after a certain rearrangement of its terms diverges almost everywhere (a.e.) on $[0, 2\pi]$.

Relying on this assertion, P. L. Ul'yanov [2] showed that the condition $f(x) \in L^2$ in it can be replaced by $f(x) \in L^p$ for all $p < \infty$. In 1960 there appeared a paper by Zagorskii [4], containing a scheme of proof of A. N. Kolmogorov' s theorem. Starting from Zagorskii' s method, P. L. Ul' yanov extended this theorem to series with respect to the Walsh and Haar systems (and also for $f(x) \in L^p$ for all $p < \infty$) [6]. Further, simultaneously by us [7] and by P. L. Ul' yanov [8], A. N. Kolmogorov' s theorem was extended to an arbitrary complete orthonormal system $\{\varphi_n(x)\}$*.

Let us also note that in [8] (for a detailed proof see [9]) it is shown that in the last result, in the case when the functions of the system $\{\varphi_n(x)\}$ are uniformly bounded, the condition $f(x) \in L^2$ can likewise be replaced by $f(x) \in L^p$ for all $p < \infty$. In connection with this, P. L. Ul' yanov raised the question (see [9]) of the possibility of generalizing the indicated results to the class of continuous functions. Below we give an affirmative answer to it.

In the present note we prove a theorem which strengthens and, in a certain sense, completes the results cited above.

Theorem. Let $\{\varphi_n(x)\}$ be an arbitrary complete orthonormal system of functions in $L^2[0, 1]$. Then there exists a continuous function $f(x)$ whose Fourier series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{2}$$

after a certain rearrangement of its terms diverges a.e. on $[0, 1]$.

This result is new also for the classical systems: the trigonometric, Haar, and Walsh systems. In particular, for the trigonometric system we obtain a direct strengthening of the above-formulated theorem of A. N. Kolmogorov:

Corollary 1. There exists a continuous function $f(x)$ whose Fourier series (1), after a certain rearrangement of its terms, diverges a.e.

Let us note one more direct consequence of our theorem.

Corollary 2. For any complete orthonormal system $\{\varphi_n(x)\}$ there exists a continuous function $f(x)$ whose Fourier series (2) diverges absolutely a.e. (i.e., $\sum |c_n \varphi_n(x)| = \infty$ a.e.).

We pass to the proof of the theorem. The main idea of the proof is the same as in our preceding note [7], where this theorem was proved for $f(x) \in L^2$. The essence of this idea is that our theorem follows from its own special case: namely, when as the system $\{\varphi_n(x)\}$ one takes

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* Thereby it was shown that complete systems of unconditional convergence do not exist.

the Haar system. Therefore the proof is divided into two parts: first we prove our theorem for the Haar system, and then, following mainly ([7]), we carry out the reduction of the general case. Denote the Haar system, numbered in the usual order, by $\chi_k(x)$ $(k = 0, 1, 2, ...)$. We formulate a lemma which reduces the problem to a simpler one.

Lemma 1. *Let $\{\varphi_n(x)\}$ be a complete orthonormal system. Then, if there exists a bounded function $F(x)$ whose Fourier series with respect to this system diverges unboundedly * a.e., then there exists a continuous function $f(x)$ having the same property.*

Proof. Let $|F(x)| \leqslant 1$. Obviously, there is a sequence of sets $\{E_k\}$ such that $E_k \subset E_{k+1}$, $\mu E_k \to 1$;

$$\sup_{x \in E_k} |\varphi_k(x)| = D_k < \infty.$$

Denote by $F_n^{(s)}(x)$ a continuous function such that $|F_n^{(s)}(x)| \leqslant 1$ and, for every $1 \leq j \leq n$,

$$\left| b_j - b_j^{(s,n)} \right| < (2^{s+n} D_n)^{-1},$$

where $\{b_j\}$ and $\{b_j^{(s,n)}\}$ are the Fourier coefficients with respect to the system $\{\varphi_j(x)\}$ of the functions $F(x)$ and, respectively, $F_n^{(s)}(x)$. We construct inductively a sequence of functions $f_k(x)$ and numbers n_k; the k-th step is carried out as follows: denote

$$\sum_{i=1}^{n_{k-1}} D_i|b_i| = M_{k-1}.$$

It is not hard to see that the set of those λ_k for which the Fourier series

$$\sum a_i^{(k)}\varphi_i(x)$$

of the function $f_{k-1}(x) + \lambda_k F(x)$ diverges unboundedly a.e. has full measure. Choosing from this set an arbitrary

$$0 < \lambda_k < (2^k M_{k-1})^{-1},$$

we find a number n_k such that

$$\mu A_k \left\{ x;\ \sup_{1\le j\le n_k} \left| \sum_{i=1}^{j} a_i^{(k)}\varphi_i(x) \right| > k \right\} > 1 - \frac{1}{2^k}.$$

Put

$$f_k(x) = f_{k-1}(x) + \lambda_k F_{n_k}^{(k)}(x).$$

Now it is not hard to prove that the function

$$f(x) = \lim_{k\to\infty} f_k(x)$$

satisfies all the requirements of the lemma.

Lemma 2. *There exists a bounded function $\Psi(x)$ on $[0,1]$ whose Fourier series with respect to the Haar system, after a certain permutation of its terms, diverges unboundedly a.e. on $[0,1]$.*

In proving this lemma we shall make essential use of the ideas of Zagorskiĭ ([4]) and Ul′yanov ([9]). Let us agree on the notation. Let

$$P(x) = \sum_{k=1}^{n} a_k \Psi_k(x).$$

Denote

$$P^*(x) = \max_{1\le k\le n} \left| \sum_{i=1}^{k} a_i \Psi_i(x) \right|.$$

Fix an arbitrary natural number N. Arrange the first 2^N Haar functions as follows. Denote

$$E_j = [0,1] \setminus \bigcup_{k=1}^{j} \Delta_k, \qquad \Delta_k = \left[\frac{k-1}{2^N}, \frac{k}{2^N} \right].$$

Obviously, $\{E_j\}$ $(j = 1, 2, \dots, 2^N - 1)$ forms a system of nested sets. Now assign to each j the Haar function $\chi_{k(j)}(x)$ for which the point $j/2^N$ is the midpoint of the interval on which this function is nonzero. It is easy to see that such a correspondence, where j and $k(j)$ run through all values from 1 to $2^N - 1$, is one-to-one. Finally, put

$$\chi_{k(j)} \equiv \Psi_j(x)^{**}.$$

Consider now the function

$$T_N(x) = \sum_{k=1}^{N} r_k(x),$$

where $\{r_k(x)\}$ is the system of Rademacher functions. It is easy to see that

$$T_N(x) = \sum_{j=1}^{2^N-1} \Psi_j(x) \int_{E_j} \Psi_j(x)\, dx.$$

* That is, its partial sums are unbounded.

** $k(0)$ is taken to be zero.

In view of the lacunarity of the Rademacher system, the inequality

$$\int_0^1 |T_N(x)|\, dx \ge \frac{1}{8}\sqrt{N} \tag{3}$$

holds (see ([10]), p. 154). We now introduce the following function: $P_N(x) = \operatorname{sign} T_N(x)$. Since it is constant on each Δ_j, it is a polynomial in the first 2^N Haar functions, and hence

$$P_N(x) = \sum_{k=1}^{2^N-1} a_k \Psi_k(x).$$

Denote the partial sums of the last polynomial by $s_j(x)$ $(0 \le j \le 2^N - 1)$. Estimating, and denoting by $\chi(E)$ the characteristic function of the set E, we have

$$\begin{aligned}\int_0^1 P_N^*(x)\,dx &\ge \int_0^1 \sum_{j=1}^{2^N-1} s_j(x)\chi(\Delta_{j+1}) \\ &= \int_0^1 \sum_{k=1}^{2^N-1} a_k \Psi_k(x) \left[\sum_{j=k+1}^{2^N} \chi(\Delta_j)\right] dx \\ &= \sum_{k=1}^{2^N-1} a_k \int_{E_k} \Psi_k(x)\,dx = \int_0^1 P_N(x) T_N(x)\,dx = \int_0^1 |T_N(x)|\,dx.\end{aligned} \tag{4}$$

Furthermore, obviously,

$$P_N^*(x) \le \sum_{k=1}^{2^N-1} |a_k \Psi_k(x)| \le N+1 \tag{5}$$

(since at each point, among the first 2^N Haar functions only $N+1$ are different from 0). Put $L_N(y) = \mu E_N\{x;\ P_N^*(x) \ge y\}$. Taking (3), (4), (5) into account, we obtain

$$\frac{1}{8}\sqrt{N} \le \int_0^1 P_N^*(x)\,dx = \int_0^\infty L_N(y)\,dy = \int_0^1 L_N(y)\,dy + \int_1^{N+1} yL_N(y)\frac{dy}{y} \le$$

$$\le 1 + \ln(N+1) \max_{1\le y\le N+1} [yL_N(y)].$$

Thus, for some y_N we have

$$y_N L_N(y_N) \ge \frac{\frac{1}{8}\sqrt{N} - 1}{\ln(N+1)} \ge CN^{1/4}.$$

It is now not difficult to construct a nondecreasing sequence of natural numbers N_k and a sequence $\gamma_k \to \infty$ such that

$$\sum_{k=1}^{\infty} \frac{\gamma_k}{y_{N_k}} < \infty; \qquad \sum_{k=1}^{\infty} \frac{N_k^{1/4}}{y_{N_k}} = \infty.$$

On the basis of Lemma 7.2 of the work ([9]), choose a sequence $\{\nu_k\}$ so that $\nu_{k+1} > N_k \nu_k$ and $\mu\left(\overline{\lim}_{k\to\infty} E_k^{(\nu_k)}\right) = 1$, where

$$E_k^{(\nu_k)} = \left\{ y;\ y = \frac{x+i}{\nu_k},\ i = 0, 1, \dots, \nu_k - 1;\ x \in E_k \right\}.$$

Now it is already clear that the function

$$\Psi(x) = \sum_{j=1}^{\infty} \frac{\gamma_k}{y_{N_k}} P_{N_k}(\nu_k x)$$

satisfies all the requirements of the lemma. Lemma 2 is proved.

In the remaining part of the proof we shall follow the presentation of our note ([7]). Let us prove a lemma analogous to Lemma 4 of ([7]).

Lemma 3. *Let an arbitrary increasing sequence $\Lambda = \{i_1, i_2, \dots\}$ be fixed. Then there exists a bounded function*

$$F(x) \sim \sum_{k=1}^{\infty} \sum_{m=2^{i_k}}^{2^{i_{k+1}}-1} b_m \chi_m(x), \tag{6}$$

whose Fourier series (6), after a certain rearrangement of the terms in the outer sum, diverges unboundedly a.e. on $[0, 1]$.

Proof. Define for the sequence Λ the sequence of functions $F(\Lambda) = \{f_k(x)\}$, as was done in ([7]) (formula (4)). Consider the series $\sum a_k f_k(x)$, where a_k are the Fourier coefficients with respect to the Haar system of the function $\Psi(x)$ constructed in the preceding lemma. By what was proved in that lemma, taking into account Lemma 3 ([7]) (modified in the natural way for the case of unbounded divergence), we obtain that the series under consideration, after some rearrangement of its terms, diverges unboundedly a.e. on $[0, 1]$. But, by formula (4) of the note ([7]),

$$a_k f_k(x) = \sum_{m=2^{i_k}}^{2^{i_{k+1}}-1} b_m \chi_m(x),$$

where b_m are certain numbers. Thus the series $\sum a_k f_k(x)$ can be represented in the form (6). It is clear that its sum, understood in the sense of convergence in the mean, is a bounded function. This fact follows immediately from our "mapping lemma" ([7]), Lemma 2), if one takes into account that the function $\Psi(x)$ is bounded. Thus Lemma 3 is proved.

We now complete the proof of the theorem. Let an arbitrary complete orthonormal system $\{\varphi_n(x)\}$ be fixed. Put

$$\int_0^1 \varphi_n(x)\chi_m(x)\,dx = a_{nm}.$$

By induction it is easy to construct increasing sequences $\Lambda = \{i_k\}$ and $\{n_k\}$ so that

$$\sum_{n=n_k}^{n_{k+1}-1} \sum_{m\in\bigcup_{s\neq k}[2^{i_s},\,2^{i_{s+1}})} a_{nm}^2 + \sum_{m=2^{i_k}}^{2^{i_{k+1}}-1} \sum_{n\in[0,n_k)+[n_{k+1},\infty)} < \frac{1}{2^k}.$$

Denote

$$c_n = \int_0^1 F(x)\varphi_n(x)\,dx,$$

where $F(x)$ is the function corresponding to Λ by Lemma 3. With the help of simple estimates we obtain:

$$\left\| \sum_{n=n_k}^{n_{k+1}-1} c_n\varphi_n(x) - \sum_{m=2^{i_k}}^{2^{i_{k+1}}-1} b_m\chi_m(x) \right\|_{L^2} < \frac{1}{2^{k-1}},$$

whence, without difficulty, taking Lemma 3 into account, we obtain that the series $\sum c_n\varphi_n(x)$, after some rearrangement of its terms, diverges unboundedly a.e. on $[0,1]$. But this series is the Fourier series with respect to the system $\{\varphi_n(x)\}$ of the bounded function $F(x)$. Taking Lemma 1 into account, we see that the theorem is completely proved.

Remark 1. The theorem can be strengthened by replacing in it the divergence of the series (2) after rearrangement by nonsummability by any preassigned method T (T^*). This fact follows directly from the theorem just proved, if one applies to it Theorem 5 of our work ([3]).

Remark 2. Using the method of the note ([5]), one can show the definitive nature of the theorem proved by us (see on this subject ([7]), § 3).

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Note: Figure translations are in progress. See original paper for figures.