ON THE PROPERTIES OF MULTIDIMENSIONAL SINGULAR INTEGRALS IN THE SPACE \(L_p(S;\rho)\)

1961

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Abstract

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MATHEMATICS

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ON THE PROPERTIES OF MULTIDIMENSIONAL SINGULAR INTEGRALS IN THE SPACE $L_p(S;\rho)$

(Presented by Academician N. I. Muskhelishvili on 6 III 1961)

1. Let E_{m+1} be Euclidean space; $x(x_1, \dots, x_{m+1})$, $y(y_1, \dots, y_{m+1})$, $z(z_1, \dots, z_{m+1}), \dots$ are points of this space. Denote by $r(x, y)$ the distance between the points x and y, and by $C(z; \delta)$ the ball with center at the point z and radius δ. Let l be an arbitrary line passing through the point z, and let δ be an arbitrary positive number. Denote by $H(l; \delta)$ the circular cylinder of height 2δ, whose axis is l, whose center of symmetry is the point z, and whose base radius is equal to δ.

Let S be a bounded, closed or open m-dimensional manifold from the space E_{m+1} (the set S will be regarded as closed), possessing the following property: to each point z of the manifold S one can assign a positive number δ and a certain rectangular coordinate system $(X_1, \dots, X_{m+1})$ with origin at the point z in such a way that the part of S contained inside $H(X_{m+1}; \delta)$ admits a representation of the form $\eta_{m+1} = \gamma(\eta_1, \dots, \eta_m)$, where $\eta_1, \dots, \eta_{m+1}$ are the coordinates of the point $y \in S \cap H(X_{m+1}; \delta)$ in the system $(X_1, \dots, X_{m+1})$; γ is a single-valued function defined on $\tau(z; \delta)$, and $\tau(z; \delta)$ is the intersection of the ball $C(z; \delta)$ with the hyperplane passing through z and perpendicular to the axis X_{m+1}. Moreover, we shall assume that there exist continuous partial derivatives of first order of the function γ in the domain $\tau(z; \delta)$, and that

$$\gamma(0, \dots, 0) = \partial\gamma(0, \dots, 0)/\partial\eta_k = 0 \quad (k = 1, \dots, m)$$

and $\omega(\partial\gamma/\partial\eta_k; t)t^{-1}$ $(k = 1, \dots, m)$ are integrable on the interval $(0, 1)$, where $\omega(\partial\gamma/\partial\eta_k; t)$ is the modulus of continuity of the function $\partial\gamma/\partial\eta_k$ in the domain $\tau(z; \delta)$.

Let on S be given an $(m+1)$-dimensional symmetric matrix $\|A_{ij}(y)\|$, possessing the following properties: the quadratic form $\sum A_{ij}(y)t_i t_j$ is positive definite at each point y of the manifold S, and the functions $\omega(A_{ij}; t)t^{-1}$ $(i, j = 1, \dots, m+1)$ are integrable on the interval $(0, 1)$.

Consider the operator

$$K_\varphi(x) = \int_S^* k(y, x)\varphi(y)\, dS_y,$$

where

$$k(y,x) = \prod_{i=1}^{m+1} (x_i - y_i)^{\lambda_i} \, [\sigma(y,x)]^{-\lambda}, \qquad \sigma^2(y,x) = \sum_{i,j=1}^{m+1} A_{ij}(y)(x_i - y_i)(x_j - y_j),$$

λ_i $(i = 1, \dots, m+1)$ and λ are arbitrary nonnegative integers;

$$\lambda - m = \sum \lambda_i$$

is a positive odd number; $\varphi(y)$ is the density of the integral; dS_y is the element of area of the manifold S at the point y, and the integral

is understood in the sense of the principal value:

$$\int_S^* k(y,x)\varphi(y)\, dS_y = \lim_{\delta \to 0} \int_{S(\delta;x)} k(y,x)\varphi(y)\, dS_y,$$

where $S(\delta; x) = S - S(x; \delta)$; $S(x; \delta) = S \cap H(n(x); \delta)$; $n(x)$ is the normal to the manifold S at the point x.

In the present article we consider the question of the boundedness of the operator K_φ in the space $L_p(S; \rho)$, where $p > 1$, and $\rho(x)$ is a certain nonnegative measurable function on S. This question, under several other assumptions, was studied in the works ($^{1-7}$). In the article a formula is derived for changing the order of integration in repeated singular integrals of the indicated type. Analogous formulas were obtained in ($^{1,8-11}$). All these questions in the case of a singular integral of Cauchy type were studied in (^{12,13}).

2. Theorem 1. If $\varphi(y) \in L_p(S)$, then K_φ is a bounded operator mapping $L_p(S)$ into itself.

Proof. It is sufficient to show the boundedness of the operator

$$T_\varphi(x) = \int_{S(z;\delta)}^* k(y,x)\varphi(y)\, dS_y$$

on $L_p(S(z, \nu))$, where z is an arbitrary point of the manifold S, $0 < \nu < \delta$, and δ is the positive constant occurring in the definition of S. Denote by $\xi_1, \dots, \xi_{m+1}$ and $\eta_1, \dots, \eta_{m+1}$ the coordinates of the points x and y in the system $(X_1, \dots, X_{m+1})$, and by ξ and η the points $(\xi_1, \dots, \xi_m, 0)$ and $(\eta_1, \dots, \eta_m, 0)$. We shall have

$$x_i = z_i + \sum_{j=1}^{m+1} a_{ij}\xi_j, \qquad \sigma^2(y,x) = \sum_{i,j=1}^{m+1} B_{ij}(y)(\xi_i - \eta_i)(\xi_j - \eta_j),$$

$$B_{ij}(y) = \sum_{k,l=1}^{m+1} A_{kl}(y) a_{ki} a_{lj} \qquad (i, j = 1, \dots, m-1). \tag{1}$$

It can be shown that the operator T_φ is represented in the form of a finite sum of operators with certain bounded coefficients of the following two types:

$$N_\varphi(x) = \int_{S(z;\delta)} n(x,y)\varphi(y)\,dS_y, \qquad M_\varphi(x) = \int^*_{\tau(z;\delta)} \frac{m(\xi,\eta)\varphi_*(\eta)}{r^m(\xi,\eta)}\,d\eta,$$

where

$$|n(x,y)| \le r^{-m}(x,y)F(r(x,y));$$

the function $F(t)t^{-1}$ is integrable on $(0,\delta)$; $\varphi_*(\eta) = \varphi(y)$; $d\eta$ is the element of area of the tangent hyperplane at the point z of the manifold S,

$$m(\xi,\eta) = \prod_{i=1}^{m} \left[\frac{\xi_i - \eta_i}{r(\xi,\eta)}\right]^{\nu_i} \left[\sum_{i,j=1}^{m} C_{ij}(\xi)\frac{\xi_i-\eta_i}{r(\xi,\eta)}\frac{\xi_j-\eta_j}{r(\xi,\eta)}\right]^{-\lambda/2};$$

the constants ν_i $(i = 1,\dots,m)$ satisfy the same conditions as the λ_i, and

$$C_{ij}(\xi) = B_{ij}(x) + 2B_{m+1,i}(x)\frac{\partial\gamma(\xi)}{\partial\xi_j} + B_{m+1,m+1}(x)\frac{\partial\gamma(\xi)}{\partial\xi_i}\frac{\partial\gamma(\xi)}{\partial\xi_j}.$$

The boundedness of the operator N_φ is obvious, and the boundedness of M_φ follows from the works of S. G. Mikhlin (^{1,2}) and A. Calderón and A. Zygmund (^{3,4}).

We note that Theorem 1 remains valid also in the case when

$$k(y,x) = \sum_{k=1}^{n} C_k(y) \prod_{i=1}^{m+1} (x_i - y_i)^{\lambda_{ik}} (\sigma_{1k} + \sigma_{2k})^{-n_k} \sigma_{1k}^{-m_k} \sigma_{2k}^{-l_k}, \tag{2}$$

where n is a natural number; $C_k(y)$ $(k = 1,\dots,n)$ are continuous functions on S; λ_{lk}, n_k, m_k, and l_k are nonnegative integers; $n_k + m_k + l_k - m = \lambda_{1k} + \dots + \lambda_{m+1,k}$ $(k = 1,\dots,n)$ are positive even numbers, and $\sigma_{1k} = \sigma_{1k}(y,x)$ and $\sigma_{2k} = \sigma_{2k}(y,x)$ $(k = 1,\dots,n)$ are expressions defined in the same way as $\sigma = \sigma(y,x)$.

Theorem 2. If $\varphi(y) \in L_p(S;\rho(y))$, where

$$\rho(y) = \prod_{k=1}^{n} r^{\alpha_k}(y, z^{(k)}),$$

$$0 < \alpha_k < m(p-1) \quad (k = 0,\dots,n_1 \le n); \qquad 0 < -\alpha_k < m \quad (k = n_1+1,\dots,n);$$

$$z^{(k)} \in S, \quad z^{(i)} \neq z^{(j)} \ (i \neq j;\ k,i,j = 1,\dots,n); \qquad p > 1,$$

then the operator K_φ maps the space $L_p(S;\rho(y))$ into itself and is a bounded operator.

Proof. It is enough to prove the validity of the theorem in two cases: 1) when $\rho(y) = r^{-\alpha}(y,z)$, where $0 < \alpha < m,\ z \in S$, and 2) when $\rho(y) = r^{\alpha}(y,z)$, where $0 < \alpha < m(p-1),\ z \in S,\ p > 1$.

Consider case 1). We have

$$|K_\varphi(x)| \leq c\left(K_f^{(1)}(x) + K_f^{(2)}(x)\right) + r^{\alpha/p}(x,z)|K_f(z)|,$$

where

$$f(x) = \varphi(x) r^{-\alpha/p}(x,z) \in L_p(S);$$

c is some positive constant;

$$K_f^{(1)}(x) = \int_S |k(y,x)|\, r^{\beta}(y,x)\, r^{\delta}(y,z)\, |f(y)|\, dS_y,$$

$$K_f^{(2)}(x) = r^{\delta}(x,z) \int_S |k(y,x)|\, r^{\beta}(y,x)\, |f(y)|\, dS_y,$$

$$\beta = -\alpha/p, \quad \delta = 0, \text{ if } \alpha/p < 1, \qquad \beta = -1, \quad \delta = \alpha/p - 1, \text{ if } \alpha/p > 1.$$

Using the inequalities given in the notes ([6],[11]), one can show that

$$\int_S r^{-\alpha}(y,z)\, |K_f^{(i)}(y)|^p\, dS_y < c \int_S |f(y)|^p\, dS_y \qquad (i = 1,2).$$

From this inequality and from Theorem 1 the proof of Theorem 2 in case 1) follows directly. Case 2) is considered analogously.

Theorem 3. If $\varphi(y) \in L_p(S;\mu(y))$, where $p > 1$,

$$\mu(y) = \prod_{k=1}^{n} r^{\gamma_k}(y, z^{(k)}),$$

$$\gamma_k = \alpha_k(p-1) \quad (k = 1,\dots,n_1 \leq n), \qquad \gamma_k = -\alpha_k \quad (k = n_1+1,\dots,n),$$

$$0 < \alpha_k < m, \quad z^{(k)} \in S, \quad z^{(i)} \neq z^{(j)} \ (i \neq j;\ k, i, j = 1, \dots, n),$$

then the operator

$$L_\varphi(x) = \frac{1}{\mu_*(x)} \int_S k(y, x) \mu_*(y) \varphi(y) \, dS_y,$$

where

$$\mu_*(y) = \prod_{k=1}^{n} r^{\omega_k}(y, z^{(k)}),$$

$$\omega_k = \alpha_k \quad (k = 1, \dots, n_1); \qquad \omega_k = -\alpha_k \quad (k = n_1 + 1, \dots, n),$$

maps the space $L_p(S; \mu(y))$ into itself and is a bounded operator.

Let us note that Theorem 2 remains valid if

$$\rho(y) = \prod_{k=1}^{n} r^{\alpha_k}(y, z^k) \lg^p\left(2dr^{-1}(y, z^{(k)})\right), \tag{3}$$

where α_k $(k = 1, \dots, n)$ satisfy the conditions of Theorem 2, $d = \operatorname{div} S$, and p_k are arbitrary integers. This assertion is proved, just like Theorem 2, with the aid of the inequalities given in note (^{14}). Theorem 3 admits an analogous generalization.

Theorems 2 and 3, as well as their generalizations just indicated, are also valid in the case of $k(y, x)$ defined by formula (2). Moreover, in the proofs of Theorems 2 and 3 only the fact is used that K_φ is a bounded operator in $L_p(S)$ and $k(y, x) = O(r^{-m}(x, y))$.

3. Let

$$k^{(i)}(y, x) = [\sigma(y, x)]^{-m-1} \sum_{k=1}^{m+1} C_k^{(i)}(y)(x_k - y_k) \qquad (i = 1, 2),$$

where $C_k^{(i)}(y)$ $(k = 1, \dots, m + 1)$ are continuous functions on S, and $B(z) = \det \|B_{ij}(z)\|_{i,j=1}^{m}$, where $B_{ij}(z)$ are defined by formula (1). Denote by $b_{ij}(z)$ the ratio of the algebraic complement of the element $B_{ij}(z)$ in the determinant B to B itself, and by $d_j^i(z)$ the expression $\sum C_k^{(i)}(z) a_{kj}$, where a_{kj} are defined from (1).

Theorem 4. If $\varphi(y) \in L_p(S; \rho(y))$, where $\rho(y)$ is a function participating in Theorem 2 or defined by formula (3), then

$$\int_S k^{(1)}(y,z)\,dS_y \int_S k^{(2)}(x,y)\varphi(x)\,dS_x = \int_S \varphi(x)\,dS_x \int_S k^{(1)}(y,z)k^{(2)}(x,y)\,dS_y -$$

$$-\frac{\Gamma^{m+1}\sum\limits_{i,j=1}^{m} b_{ij}(z)d_i^{(1)}(z)d_j^{(2)}(z)}{m\Gamma^2\left(\dfrac{m+1}{2}\right)B(z)}\,\varphi(z).$$

4. The theorems formulated make it possible to study integral equations containing the indicated singular integrals extended over open manifolds.

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Received
3 III 1961

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