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Abstract

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ON THE CONDITIONS FOR SOLVABILITY OF A CERTAIN BOUNDARY-VALUE PROBLEM FOR A PARABOLIC SYSTEM

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1. Consider the system of differential equations

$$\frac{\partial u_i}{\partial t} = \sum_{k=1}^{2} a_{ik} \Delta u_k, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad t > 0, \tag{1}$$

where a_{ik} are real constants satisfying the following conditions: all the roots of the equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \tag{2}$$

are distinct and positive; denote them by λ_1, λ_2. We shall seek a solution of equation (1) in the domain D $(0 < x < l,\ -\infty < y < +\infty)$, satisfying the initial condition

$$u_i\big|_{t=0} = 0 \tag{3}$$

and the boundary conditions

$$\begin{aligned} (\alpha_1 u_1 + \alpha_2 u_2)\big|_{x=0} &= \psi_1(y,t), \quad (\partial u_1/\partial x + h_1 u_1)\big|_{x=0} = \psi_2(y,t), \\ (\beta_1 u_1 + \beta_2 u_2)\big|_{x=l} &= \psi_3(y,t), \quad (\partial u_2/\partial x + h_2 u_2)\big|_{x=l} = \psi_4(y,t), \end{aligned} \tag{4}$$

where α_i, β_i, h_i are prescribed constant quantities; $\psi_i(y,t)$ are known continuous bounded functions with their derivatives of sufficiently high order, and $\psi_i(y,0) = 0$. Many more general boundary-value problems are reduced, by means of linear transformations, to the one under consideration.

2. Introduce the following notation:

$$\int_0^t d\tau \int_{-\infty}^{+\infty} G(x, y-\eta, t-\tau)\Phi(\eta,\tau)\, d\eta = G * \Phi\,[x,y,t]. \tag{5}$$

We shall seek the solution of this problem in the following form (^{1}):

$$\begin{aligned} u_1(x,y,t) &= \sum_{i,j=1}^{2} A_{1j}^i g_x^j * \omega_{1i}[x,y,t] + \sum_{i,j=1}^{2} A_{1j}^i g_x^j * \omega_{2i}[l-x,y,t], \\ u_2[x,y,t] &= \sum_{i,j=1}^{2} A_{2j}^i g_x^j * \omega_{1i}[x,y,t] + \sum_{i,j=1}^{2} A_{2j}^i g_x^j * \omega_{2i}[l-x,y,t], \end{aligned} \tag{6}$$

where

$$g^j(x,y,t) = \frac{1}{2\pi t}\exp\left[-\frac{x^2+y^2}{4\lambda_j t}\right], \qquad g_x^j = \frac{\partial}{\partial x} g^j, \qquad g_0^j = g^j(0,y,t); \tag{7}$$

$$\begin{aligned} &A_{11}^1 = \frac{\lambda_1 - a_{22}}{\lambda_1-\lambda_2}, \quad A_{12}^1 = \frac{a_{22}-\lambda_2}{\lambda_1-\lambda_2}, \quad A_{11}^2 = \frac{a_{12}}{\lambda_1-\lambda_2}, \quad A_{12}^2 = -\frac{a_{12}}{\lambda_1-\lambda_2}, \\ &A_{21}^1 = \frac{a_{21}}{\lambda_1-\lambda_2}, \quad A_{22}^1 = -\frac{a_{21}}{\lambda_1-\lambda_2}, \quad A_{21}^2 = \frac{\lambda_1 - a_{11}}{\lambda_1-\lambda_2}, \quad A_{22}^2 = \frac{a_{11}-\lambda_2}{\lambda_1-\lambda_2}, \end{aligned} \tag{8}$$

$$\sum_{j=1}^{2} A_{ij}^k = \begin{cases} 0, & i \neq k, \\ 1, & i = k; \end{cases} \qquad \sum_{i=1}^{2} a_{\alpha i} A_{ij}^k = \lambda_j A_{\alpha j}^k.$$

By direct verification it is established that the functions defined by equality (7) satisfy system (1) and the initial condition (3). It remains to determine ω so that the functions (7) satisfy the boundary conditions (4).

3. We first state the following lemma:

Lemma. *If $\omega(y,t)$ has bounded derivatives $\partial\omega/\partial t$, $\partial^2\omega/\partial y^2$, and $\omega(y,0)=0$, then*

$$\lim_{x\to 0}\int_0^t d\tau \int_{-\infty}^{+\infty} g_{xx}^j(x, y-\eta, t-\tau)\,\omega(\eta,\tau)\, d\eta = \frac{1}{\lambda_j} g^j * F_j[\omega]\,[0,y,t], \tag{9}$$

where

$$F_j[\omega] = \partial\omega/\partial\tau - \lambda_j \partial^2\omega/\partial\eta^2.$$

The lemma is proved by integration by parts after replacing g^j_{xx} by

$$-\frac{1}{\lambda_j} g^j_\tau - g^j_{\eta\eta}.$$

To determine the functions $\omega_{ij}(y,t)$, let us substitute (6) into (4). In doing so, using the lemma and the property of the "potential" of a double layer ([1], [2]), we obtain the following system of integro-differential equations:

$$-\alpha_1\omega_{11} - \alpha_2\omega_{12} + \sum_{i,j=1}^{2} \left(\alpha_1 A^i_{1j} + \alpha_2 A^i_{2j}\right) g^j_x * \omega_{2i}\,[l,y,t] = \psi_1(y,t); \tag{10}$$

$$\begin{aligned} &\sum_{i,j=1}^{2} A^i_{1j} \frac{1}{\lambda_j} g^j * F_j[\omega_{1i}]\,[0,y,t] + \sum_{i,j=1}^{2} A^i_{2j} g^j_{xx} * \omega_{2i}\,[l,y,t] \\ &- h_1\omega_{11}(y,t) + h_1 \sum_{i,j=1}^{2} A^i_{1j} g^j_x * \omega_{2i}\,[l,y,t] = \psi_2(y,t); \end{aligned} \tag{11}$$

$$-\beta_1\omega_{21} - \beta_2\omega_{22} + \sum_{i,j=1}^{2} \left(\beta_1 A^i_{1j} + \beta_2 A^i_{2j}\right) g^j_x * \omega_{1i}\,[l,y,t] = \psi_3(y,t); \tag{12}$$

$$\begin{aligned} &\sum_{i,j=1}^{2} A^i_{2j} \frac{1}{\lambda_i} g^j * F_j[\omega_{2i}]\,[0,y,t] + \sum_{i,j=1}^{2} A^i_{2j} g^j_{xx} * \omega_{1i}\,[l,y,t] \\ &- h_2\omega_{22}(y,t) + g_2 \sum_{i,j=1}^{2} A^i_{2j} g^j_x * \omega_{1i}\,[l,y,t] = \psi_4(y,t). \end{aligned} \tag{13}$$

Eliminating the function ω_{12} from equations (10), (11), we obtain

$$\begin{aligned} &\sum_{j=1}^{2} \left(\alpha_2 A^1_{1j} - \alpha_1 A^2_{1j}\right) \frac{1}{\lambda_j} g^j * F_j[\omega_{11}]\,[0,y,t] \\ &= \alpha_2 h_1 \omega_{11}(y,t) - \sum_{i=1}^{2} K^i_1 * \omega_{2i}\,[l,y,t] + \varphi_2(y,t), \end{aligned} \tag{14}$$

where

$$
\begin{aligned}
K_1^i(l, y-\eta, t-\tau) = {} & \sum_{\nu,j=1}^{2} A_{1j}^2 \left(\alpha_1 A_{1\nu}^i + \alpha_2 A_{2\nu}^i\right) \frac{1}{\lambda_j} F_j[g_x^i] * g^j \left[l, y-\eta, t-\tau\right] \\
& + \sum_{\nu,j=1}^{2} \alpha_2 A_{1j}^\nu g_{xx}^j (l, y-\eta, t-\tau) + h_1 \sum_{\nu,j=1}^{2} \alpha_2 A_{1j}^\nu g_x^j (l, y-\eta, t-\tau),
\end{aligned}
\tag{15}
$$

$$
\varphi_2(y,t) = \alpha_2 \psi_2 + \sum_{j=1}^{2} A_{1j}^2 \frac{1}{\lambda_j} g^j * F_j[\psi_1]\,[0, y, t]. \tag{16}
$$

Eliminating ω_{21} from (12) and (13), we have

$$
\begin{aligned}
& \sum_{j=1}^{2} \left(\beta_2 A_{2j}^1 - \beta_1 A_{2j}^2\right) \frac{1}{\lambda_j} g^j * F_j[\omega_{22}]\,[0, y, t] \\
& = -\beta_1 h_2 \omega_{22} + \sum_{i=1}^{2} K_2^i * \omega_i [l, y, t] - \varphi_4(y,t),
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
K_2^i(l, y-\eta, t-\tau) = {} & \sum_{\nu,j=1}^{2} A_{2j}^1 (\beta_1 A_{1\nu}^i + \beta_2 A_{2\nu}^i) \frac{1}{\lambda_j} F_j[g_x^\nu] * g^j [l, y-\eta, t-\tau] \\
& + \sum_{\nu,j=1}^{2} \beta_1 A_{2j}^\nu g_{xx}^j (l, y-\eta, t-\tau) + h_2 \sum_{\nu,j=1}^{2} \beta_1 A_{2j}^\nu g_x^j (l, y-\eta, t-\tau),
\end{aligned}
\tag{18}
$$

$$
\varphi_n(y,t) = \beta_1 \varphi_4(y,t) + \sum_{j=1}^{2} A_{2j}^2 \frac{1}{\lambda_j} g^j * F_j[\psi_3][0, y, t]. \tag{19}
$$

It is obvious that the kernels K_1^i and K_2^i are regular.

4. Instead of equations (14) and (17), we shall consider their characteristic equation

$$
F[\omega] = A_1 \frac{1}{\lambda_1} g^1 * F_1[\omega][0, y, t] + A_2 \frac{1}{\lambda_2} g^2 * F_2[\omega][0, y, t] = f(y,t). \tag{20}
$$

Assuming that the Fourier-Laplace transforms can be applied to the functions ω and f, we apply it to both sides of (20)*. Then

$$\tilde{\omega} = \frac{\sqrt{\lambda_1 \lambda_2}}{A_1 \sqrt{\lambda_2} \sqrt{p + \lambda_1 s^2} + A_2 \sqrt{\lambda_1} \sqrt{p + \lambda_2 s^2}} \tilde{f}. \tag{21.1}$$

If $A_1^2 \lambda_2 - A_2^2 \lambda_1 \neq 0$, then

$$\tilde{\omega} = \frac{\sqrt{\lambda_1 \lambda_2}}{A_1^2 \lambda_2 - A_2^2 \lambda_1} \frac{A_1 \sqrt{\lambda_2} \sqrt{p + \lambda_1 s^2} - A_2 \sqrt{\lambda_1} \sqrt{p + \lambda_2 s^2}}{p + A s^2} \tilde{f}, \tag{21.2}$$

where

$$A = \frac{\lambda_1 \lambda_2 (A_1^2 - A_2^2)}{A_1^2 \lambda_2 - A_2^2 \lambda_1} = \lambda_1 \lambda_2 \frac{A_1 + A_2}{A_1 \sqrt{\lambda_2} + A_2 \sqrt{\lambda_1}} \frac{A_1 - A_2}{A_1 \sqrt{\lambda_2} - A_2 \sqrt{\lambda_1}}.$$

Let $A \geq 0$; then**

$$\begin{aligned} &\frac{\sqrt{\lambda_1 \lambda_2}\, p}{A_1 \sqrt{\lambda_2} \sqrt{p + \lambda_1 s^2} + A_2 \sqrt{\lambda_1} \sqrt{p + \lambda_2 s^2}} \doteq \frac{A_1 \lambda_2}{\sqrt{2\pi} (A_1^2 \lambda_2 - A_2^2 \lambda_1)} \frac{\exp[-y^2/4\lambda_1 t]}{t} \\ &- \frac{A_2 \lambda_1}{\sqrt{2\pi} (A_1^2 \lambda_2 - A_2^2 \lambda_1)} \frac{\exp[-y^2/4\lambda_2 t]}{t} \\ &- \frac{(\lambda_1 - A) A_1 \lambda_1 \lambda_2^{3/2}}{\sqrt{2\pi} (A_1^2 \lambda_2 - A_2^2 \lambda_1)} \int_0^1 \frac{\exp[-y^2/4a_1^2(z)t]}{t\sqrt{z}\, a_1^3(z)} \left[1 - \frac{y^2}{2a_1^2(z)t}\right] dz \\ &+ \frac{(\lambda_2 - A) A_2 \lambda_1^{3/2} \lambda_2}{\sqrt{2\pi} (A_1^2 \lambda_2 - A_2^2 \lambda_1)} \int_0^1 \frac{\exp[-y^2/4a_2^2(z)t]}{t\sqrt{z}\, a_2^3(z)} \left[1 - \frac{y^2}{2a_2^2(z)t}\right] dz \\ &= \Gamma(0, y, t), \end{aligned} \tag{22}$$

where $a_i(z) = \sqrt{\lambda_i z + A(1 - z)}$.

If $A < 0$, then

$$\begin{aligned} &\frac{\sqrt{\lambda_1 \lambda_2 p}}{A_1 \sqrt{\lambda_2} \sqrt{p + \lambda_1 s^2} + A_2 \sqrt{\lambda_1} \sqrt{p + \lambda_2 s^2}} \doteq \frac{(\lambda_2 - \lambda_1) \lambda_1 \lambda_2 (A_2 |A_1| - A_1 |A_2|) s}{(A_1^2 \lambda_2 - A_2^2 \lambda_1)^2} e^{-A s^2 t} \\ &+ \frac{\sqrt{\lambda_1 \lambda_2} A_1}{2\sqrt{\pi} (A_1^2 \lambda_2 - A_2^2 \lambda_1)} \int_t^\infty \frac{e^{-[\lambda_1 \tau + A(t - \tau)] s^2}}{\tau^{3/2}} d\tau \\ &- \frac{\sqrt{\lambda_2 \lambda_1} A_2}{2\sqrt{\pi} (A_1^2 \lambda_2 - A_2^2 \lambda_1)} \int_t^\infty \frac{e^{-[\lambda_2 \tau + A(t - \tau)] s^2}}{\tau^{3/2}} d\tau. \end{aligned} \tag{23}$$

* The Fourier transform may be applied to generalized functions, assuming that these functions grow (3).

** $\doteq$ means that one should first find the original, and then apply the inverse Fourier transform.

It follows from this that if $A_1/A_2 < 0$, then, in view of the presence of the first term in (23), the inverse Fourier transform in the ordinary sense does not exist. But if $A_1/A_2 > 0$, then it always exists, and moreover

$$\begin{aligned}&\frac{\sqrt{\lambda_1\lambda_2 p}}{A_1\sqrt{\lambda_2}\sqrt{p+\lambda_1 s^2}+A_2\sqrt{\lambda_1}\sqrt{p+\lambda_2 s^2}} \overset{\cdot}{\sim} \frac{\sqrt{\lambda_1}\lambda_2 A_1}{2\sqrt{2\pi}\,(A_1^2\lambda_2-A_2^2\lambda_1)}\\ &\times\int_1^\infty \frac{\exp\{-y^2/4a_1^2(z)t\}}{tz^{3/2}a_1(z)}\,dz-\frac{\lambda_1\sqrt{\lambda_2}A_2}{2\sqrt{2\pi}\,(A_1^2\lambda_2-A_2^2\lambda_1)}\int_1^\infty \frac{\exp\{-y^2/4a_2^2(z)t\}}{tz^{3/2}a_2(z)}\,dz\\ &=\Gamma(0,y,t).\end{aligned} \tag{24}$$

\end{equation}

Obviously, $\Gamma(0,y,t)$, as a kernel, is regular and has the same singularity as $g(0,y,t)$. We shall call $\Gamma(0,y,t)$ the resolvent.

Thus, as a result, we have the following theorem:

Theorem. If $A_1^2\lambda_2 - A_2^2\lambda_1 \neq 0$, then, if $A_1/A_2 < 0$, $(A_1+A_2)/(A_1\sqrt{\lambda_2}+A_2\sqrt{\lambda_1}) < 0$, equation (20) has no solution. In all other cases equation (20) is always solvable, and the function

$$\omega(y,t)=\Gamma^{-1}[f]=\int_0^t d\tau\int_{-\infty}^{+\infty}\Gamma(0,y-\eta,t-\tau)f(\eta,\tau)\,d\eta=\Gamma * f[0,y,t] \tag{25}$$

satisfies equation (20), if $f(y,t)$ has continuous bounded derivatives of the 2nd order with respect to the 1st argument and of the 1st order with respect to the 2nd.

The second part of the theorem is established by substituting (25) into (20). Thus all the conditions imposed on ω and f at the beginning of the paragraph are removed.

If $A_1^2\lambda_2 - A_2^2\lambda_1 = 0$, then under stronger restrictions on $f(y,t)$ the equation has a solution, but it is not obtained by passage to the limit from (25) as $A_1^2\lambda_2 - A_2^2\lambda_1 \to 0$. Therefore we shall call this case special.

Now let us consider equations (14) and (17), assuming that for each equation the characteristic part satisfies the solvability conditions. Let $\Gamma_1(0,y,t)$ and $\Gamma_2(0,y,t)$ be the resolvents of equations (14) and (17). Then, applying the inverse operator Γ^{-1} to both equations, we obtain

$$\omega_{11} = \alpha_2 h_1 \Gamma_1 * \omega_{11}[0,y,t] - \sum_{i=1}^{2} K_1^i * \Gamma_1 * \omega_{2i}[l,y,t] + \Gamma_1 * \varphi_2[0,y,t]; \qquad (14')$$

$$\omega_{22} = -\beta_1 h \Gamma_2 * \omega_{22}[0,y,t] + \sum_{i=1}^{2} K_2^i * \Gamma_2 * \omega_{1i}[l,y,t] + \Gamma_2 * \varphi_4'[0,y,t]. \qquad (17')$$

Now we first replace the functions ω_{11}, ω_{22} in equations (10) and (12) by the right-hand sides of (14) and (17). The equations thus obtained, together with (14) and (17), constitute a system of integral equations to which the method of successive approximations can already be applied.

Summarizing the results obtained, we may assert that our boundary-value problem has solutions only when the characteristic equations of equations (14) and (17) are simultaneously invertible. In the special case the boundary-value problem is ill-posed.

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[1] E. I. Kim, L.-P. Ivanova, DAN, **126**, No. 6 (1959).
[2] L. P. Ivanova, E. I. Kim, Izv. AN SSSR, Energetics and Automation, No. 3 (1959).
[3] E. I. Kim, DAN, **113**, No. 2 (1957).

Note: Figure translations are in progress. See original paper for figures.

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