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Abstract

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ASYMPTOTICS OF THE GREEN FUNCTION FOR SOBOLEV-GALPERN EQUATIONS

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§ 1. In the present work asymptotic formulas are obtained for the Green function (as $|x| \to \infty$ and fixed $t > 0$) for the Cauchy problem posed for Sobolev-Galpern equations with one spatial variable, i.e., equations of the form

$$P(\partial/\partial t,\ i\partial/\partial x)u = 0, \tag{1}$$

for which

$$\operatorname{Re}\lambda_j(\sigma) < c, \quad j = 1, 2, \dots, l, \tag{2}$$

where $P(\lambda, s)$ is a polynomial in two variables with constant coefficients; $s = \sigma + i\tau$; $\lambda_j(s)$ are the roots of the equation

$$P(\lambda, s) = P_l(s)\lambda^l + P_{l-1}(s)\lambda^{l-1} + \dots + P_0(s)\,\Big|_{t=0} = 0. \tag{3}$$

The Green function $G(x, t)$ is the solution of (1) with the initial conditions

$$G\,\Big|_{t=0} = \partial G/\partial t\,\Big|_{t=0} = \dots = \partial^{l-2}G/\partial t^{l-2}\,\Big|_{t=0} = 0, \quad \partial^{l-1}G/\partial t^{l-1}\,\Big|_{t=0} = \delta(x).$$

As a consequence of the asymptotic formulas, we obtain classes of existence for the solution of the Cauchy problem for equation $(1)^*$.

To obtain the asymptotics of the Green function we use the saddle-point method. For equations correct in the sense of Petrovskii, the asymptotics of the Green function by the saddle-point method were obtained by M. V. Fedoryuk (^{1}). The Cauchy problem for equations not solved with respect to the highest derivative in t was first studied by S. L. Sobolev (^{2}). The case of a general system of the indicated type with many independent variables was considered by S. A.

Galpern (^{3,4}), who obtained a solution of the Cauchy problem and a uniqueness theorem in a certain class of functions. The question of existence and uniqueness for the case of exponentially growing functions was considered in the work of A. G. Kostyuchenko and G. I. Eskin (^{5}). G. I. Eskin obtained, for equations (1) with one spatial variable, a more precise uniqueness theorem.

§ 2. Let us first consider an equation of first order in t,

$$Q(i\partial/\partial x)\,\partial u/\partial t = P(i\partial/\partial x)u. \tag{4}$$

Expand $P(s)/Q(s)$ in a series in a neighborhood of infinity:

$$P(s)/Q(s) = \alpha_n s^n + \alpha_{n-1}s^{n-1} + \cdots \tag{5}$$

By virtue of condition (2), only the following cases are possible:

$$\begin{array}{l} 1^\circ.\ \operatorname{Re}\alpha_n < 0, \quad n \text{ even}, \quad n > 0. \\ 2^\circ.\ \operatorname{Re}\alpha_n = \operatorname{Re}\alpha_{n-1} = \cdots = \operatorname{Re}\alpha_{p+1} = 0, \quad \operatorname{Re}\alpha_p < 0, \quad p > 0, \quad p \text{ even}. \\ 3^\circ.\ \operatorname{Re}\alpha_n = \operatorname{Re}\alpha_{n-1} = \cdots = \operatorname{Re}\alpha_1 = 0; \quad \text{a) } n \geq 2, \quad \text{b) } n < 2. \end{array} \tag{6}$$

Expand $P(s)/Q(s)$ in a neighborhood of the real root σ_k of the equation $Q(s) = 0$:

$$\frac{P(s)}{Q(s)} = \sum_{j=n}^{0} \frac{\alpha_{jk}}{(s-\sigma_k)^j} + \frac{P_1(s)}{Q_1(s)}. \tag{7}$$

* By classes of uniqueness, existence, and correctness we shall understand them as defined by I. M. Gel'fand

Only the following cases are possible:

$$\left.\begin{array}{l} 1^0.\ \operatorname{Re}\alpha_{n_k} < 0, \quad n_k \text{ even}. \\ 2^0.\ \operatorname{Re}\alpha_{n_k} = \ldots = \operatorname{Re}\alpha_{(p+1)k} = 0, \quad \operatorname{Re}\alpha_{p_k} < 0, \quad p_k > 0, \quad p_k \text{ even}. \\ 3^0.\ \operatorname{Re}\alpha_{n_k} = \ldots = \operatorname{Re}\alpha_{1_k} = 0. \end{array}\right\} \tag{8}$$

The asymptotics of the Green' s function for equation (4) is determined by: 1) the behavior of $P(s)/Q(s)$ in a neighborhood of the real poles; 2) the behavior of $P(s)/Q(s)$ in a neighborhood of the nonreal poles; 3) the behavior of $P(s)/Q(s)$ as $|s| \to \infty$. Accordingly, it has three terms:

$$G(x,t) \sim G_1(x,t) + G_2(x,t) + G_3(x,t). \tag{9}$$

Denote:

$$1 + O\Big(|x|^{-\frac{1}{n_k+1}}\Big) = h, \qquad 1 + O\Big(|x|^{-\frac{1}{n-1}}\Big) = g.$$

Theorem 1. A. For $G_1(x,t)$ in cases 1^0 (8), 2^0 (8), and 3^0 (8), respectively, the following formulas hold:

1^0 (8).

$$G_1(x,t) = \sum_{k=1}^{m} \Big\{ e^{-r_k} \Big\{ C_{1k}|x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \exp\Big[\gamma_{1k}|x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}} h\Big]$$

$$+ C_{2k}|x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \exp\Big[\gamma_{2k}|x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}} h\Big]\Big\}\Big\}, \tag{10}$$

where $\operatorname{Re}\gamma_{1k} < 0$, $\operatorname{Re}\gamma_{2k} < 0$.

2^0 (8). If n_k is even, then as $|x| \to \infty$

$$G_1(x,t) = \sum_{k} \Big\{ e^{-ix\sigma_k} \Big\{ C_k|x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \times$$

$$\times \exp\Big[i\gamma_{1k}|x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}} h + \gamma_{2k}|x|^{\frac{p_k}{n_k+1}} t^{\frac{n_k-p_k+1}{n_k+1}} h\Big]\Big\}\Big\}; \tag{11}$$

here $\operatorname{Re}\gamma_{2k} < 0$, γ_{1k} is real.

If n_k is odd, then $G_1(x,t)$ as $x \to \operatorname{sign}(-i\alpha_{n_k})\infty$ will be the same as in case 1^0 (8), while as $x \to \operatorname{sign}(i\alpha_{n_k})\infty$, $G_1(x,t)$ is the sum of two expressions of the form (11).

3^0 (8). If n_k is even, then as $|x| \to \infty$

$$G_1(x,t) = \sum \Big\{ e^{-ix\sigma_k} \Big\{ C_k|x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \exp\Big[i\gamma_k|x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}}\Big]\Big\}\Big\}; \tag{12}$$

here γ_k is real. If n_k is odd, then $G_1(x,t)$ as $x \to \operatorname{sign}(-i\alpha_{n_k})\infty$ will be the same as in case 1^0 (8), while as $x \to \operatorname{sign}(i\alpha_{n_k})\infty$, $G_1(x,t)$ is the sum of two expressions of the form (12).

B.

$$G_2(x,t) = \sum_{j=1}^{n} C_j |x|^{-\frac{n_j+2}{2n_j+2}} t^{\frac{1}{2n_j+2}} \exp\left\{-\left[ixs_j\left[1+O\Big(|x|^{-\frac{1}{n_j+1}}\Big)\right]\right]\right\}, \tag{13}$$

here $s_j = \sigma_j + i\tau_j$ is a complex root of the equation $Q(s) = 0$; η_j is its multiplicity, and only those roots for which $x\tau_j < 0$ are considered.

C. $G_3(x,t)$ in cases 1^0 (6), 2^0 (6), 3^0a (6), and 3^0b (6) has the form:

$$1^0\ (6).\quad G_3(x,t) = C_1|x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp\left[\gamma_1 |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}}\right] +$$

$$+ C_2|x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp\left[\gamma_2 |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}} g\right]; \tag{14}$$

here $\operatorname{Re}\gamma_1 < 0$, $\operatorname{Re}\gamma_2 < 0$.

2° (6). If n is even, then as $|x| \to \infty$

$$G_3(x,t) = C|x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp\left[i\gamma_1 |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}} g + \gamma_2 |x|^{\frac{p}{n-1}} t^{\frac{n-p+1}{n-1}} g\right]; \tag{15}$$

here γ_1 is real, $\operatorname{Re}\gamma_2 < 0$. If n is odd, then $G_3(x,t)$ as $x \to \operatorname{sign}(-i\alpha_n)\infty$ is the sum of two expressions of the form (15), while as $x \to \operatorname{sign}(i\alpha_n)\infty$, $G_3(x,t)$ will be the same as in case 1° (6).

3°a (6). If n is even, then as $|x| \to \infty$

$$G_3(x,t) = C|x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp\left[i\gamma |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}} g\right]; \tag{16}$$

where γ is real. If n is odd, then $G_3(x,t)$ as $x \to \operatorname{sign}(-i\alpha_n)\infty$ is the sum of two expressions of the form (16), while as $x \to \operatorname{sign}(i\alpha_n)\infty$, $G_3(x,t)$ will be the same as in case 1° (6).

3°b (6). In this case $G_3(x,t) \equiv 0$.

§ 3. Let us now consider the arbitrary Sobolev—Galpern equation (1). We shall assume that for every s the roots $\lambda_j(s)$ are distinct. Let $\sigma_1, \dots, \sigma_m$ be real, and let $s_1 = \sigma_1 + i\tau_1, \dots, s_p = \sigma_p + i\tau_p$ be complex poles of $\lambda_j(s)$, with only those s_j for which $x\tau_j < 0$ being taken. We expand $\lambda_j(s)$ in a Newton—Puiseux series in a neighborhood of the infinitely remote point and in a neighborhood of the poles. Let, in a neighborhood of infinity,

$$\lambda_j(s) = \alpha_0 s^{k_0} + \alpha_1 s^{k_1} + \dots + \alpha_p s^{k_p} + \cdots, \quad k_0 > k_1 > \cdots,$$

and, in a neighborhood of the poles,

$$\lambda_j(s) = \alpha_0(s-\sigma_k)^{k_0} + \alpha_1(s-\sigma_k)^{k_1} + \dots + \alpha_p(s-\sigma_k)^{k_p} + \cdots, \quad k_0 < k_1 < \cdots.$$

In both cases $\alpha_0 \neq 0$, $k_i = l_i/m$, l_i and m are integers.

For a neighborhood of infinity, only the following cases are possible: 1) $k_0 \leq 0$; 2) $k_0 > k_1 > \dots > k_m > 0,\ k_{m+1} \leq 0;\ \operatorname{Re}\alpha_0 = \dots = \operatorname{Re}\alpha_m = 0$; then $k_0, k_1, \dots, k_m$ are integers; 3) $k_0 > k_1 > \dots > k_m > 0,\ k_{m+1} \leq 0;\ \operatorname{Re}\alpha_0 = \dots = \operatorname{Re}\alpha_{p-1} = 0,\ \operatorname{Re}\alpha_p \neq 0,\ p \leq m$; then $k_0, k_1, \dots, k_p$ are integers, p is even, and $\operatorname{Re}\alpha_p < 0$.

For the expansion in a neighborhood of a real pole σ_k, only the following cases are possible: 1) $k_0 < k_1 < \dots < k_m < 0,\ k_{m+1} \geq 0;\ \operatorname{Re}\alpha_0 = \dots = \operatorname{Re}\alpha_m = 0$; then $k_0, k_1, \dots, k_m$ are integers; 2) $k_0 < k_1 < \dots < k_m < 0,\ k_{m+1} \geq 0;\ \operatorname{Re}\alpha_0 = \dots = \operatorname{Re}\alpha_{p-1} = 0,\ \operatorname{Re}\alpha_p \neq 0,\ p \leq m$; then $k_0, k_1, \dots, k_p$ are integers, p is even, and $\operatorname{Re}\alpha_p < 0$.

These expansions make it possible to reduce the problem of finding the asymptotics of the Green' s function for equation (1) to finding the asymptotics of the Green' s function for equations (4) of first order in t. Let the Newton–Puiseux series expansions of the function

$$(-1)^n \Big/ \prod_{k,\ k\neq j} [\lambda_k(s) - \lambda_j(s)]$$

begin, in a neighborhood of infinity, with the term $a_j s^{\lambda_j}$, and in a neighborhood of the poles with the term $b_{jk} s^{\lambda_{jk}}$.

Theorem 2. Let $G(x,t)$ be the Green' s function for the arbitrary equation (1). Then, as $|x| \to \infty$ and for constant $t > 0$,

$$G(x,t) \sim \sum_j G_{j_1}(x,t) + \sum_j G_{j_2}(x,t) + \sum_j G_{j_3}(x,t).$$

Here $G_{j_1}(x,t)$ depends only on the behavior of $\lambda_j(s)$ in a neighborhood of the real poles $\lambda_j(s)$; $G_{j_2}(x,t)$ depends only on the behavior of $\lambda_j(s)$ in a neighborhood of the complex poles; $G_{j_3}(x,t)$ depends only on the behavior of $\lambda_j(s)$ in a neighborhood of the infinitely remote point. $G_{j_1}(x,t)$ and $G_{j_2}(x,t)$ differ from the corresponding terms of the asymptotics of the Green' s function for equation (4) by the factor

$$b_{jk}\left(\frac{|x|}{|\alpha_{n_k}|t}\right)^{-\frac{\lambda_{jk}}{n_k+1}},$$

and $G_{j_3}(x,t)$—by the factor

$$a_j|x|^{\frac{\lambda_j}{n-1}}\left(|\alpha_n|t\right)^{-\frac{\lambda_j}{n-1}}.$$

Theorem 3. The asymptotics of the derivatives of the Green' s function for equation (1) is obtained by differentiating with respect to x and t the asymptotics of the Green' s function Γ.

§ 4. Let W be the class of existence of the solution of the Cauchy problem for equation (1). Each term $\sum_j G_{j_1}(x,t)$, $\sum_j G_{j_2}(x,t)$, and $\sum_j G_{j_3}(x,t)$ in the asymptotics of the Green' s function gives a class of functions W_i $(i=1,2,3)$, $W=\bigcap_i W_i$.

Let us describe the classes W_i.

$\varphi(x)\subset W_1$, if:

A^*. In certain neighborhoods of the real zeros of the polynomial $P_l(\sigma)$

$$\widetilde{\varphi}(x)=|P_l(\sigma)|^{l-1}\psi(\sigma),$$

where $P_l(\sigma)$ is defined by formula (3), $\psi(\sigma)$ is a functional of the type of a function, and $\widetilde{\varphi}(x)$ is the Fourier transform of the function $\varphi(x)$.

B. $\varphi(x)$ satisfies the following estimates:

1) If $\lambda(s)$ has a pole of type 1^0 (8), then

$$|\varphi(x)|<C_\varepsilon\exp\left[\varepsilon|x|^{\frac{n_k}{n_k+1}}\right]$$

for every $\varepsilon>0$.

2) If $\lambda(s)$ has a pole of type 2^0 (8) with even n_k, then

$$|\varphi(x)|<C_\varepsilon\exp\left[\varepsilon|x|^{\frac{p_k}{n_k+1}}\right]$$

for every $\varepsilon>0$.

3) If $\lambda(s)$ has a pole of type 2^0 (8) with odd n_k, then

$$|\varphi(x)|<C_\varepsilon\exp\left[\varepsilon|x|^{\frac{n_k}{n_k+1}}\right]$$

when $\operatorname{sign}(ix\alpha_{n_k})=-1$, for every $\varepsilon>0$;

$$|\varphi(x)|<C_\varepsilon\exp\left[\varepsilon|x|^{\frac{p_k}{n_k+1}}\right]$$

when $\operatorname{sign}(ix\alpha_{n_k})=1$, for every $\varepsilon>0$.

4) If $\lambda(s)$ has a pole of type 3^0 (8) with even n_k, then

$$|\varphi(x)| < C_\varepsilon \left(|x|^{\frac{2\lambda j_k - 2l n_k + n_k - \varepsilon}{2n_k+2}} + 1 \right)$$

for some $\varepsilon > 0$.

5) If $\lambda(s)$ has a pole of type 3^0 (8) with odd n_k, then

$$|\varphi(x)| < C_\varepsilon \exp\left[\varepsilon |x|^{\frac{n_k}{n_k+1}} \right]$$

when $\operatorname{sign}(ix\alpha_{n_k}) = -1$, for every $\varepsilon > 0$;

$$|\varphi(x)| < C_\varepsilon \left(|x|^{\frac{2\lambda j_k - 2l n_k + n_k - \varepsilon}{2n_k+2}} + 1 \right)$$

when $\operatorname{sign}(ix\alpha_{n_k}) = 1$, for some $\varepsilon > 0$.

If $\lambda(s)$ has poles satisfying several of the conditions listed here at once, then the class W_1 is obtained as the intersection of the corresponding classes of functions.

$\varphi(x) \subset W_2$, if:

$$|\varphi(x)| < C_\varepsilon e^{(a_1 - \varepsilon)|x|}$$

for $x < 0$, for some $\varepsilon > 0$;

$$|\varphi(x)| < C_\varepsilon e^{(a_2 - \varepsilon)|x|}$$

for $x > 0$, for some $\varepsilon > 0$, where a_1 is the distance from the poles $\lambda(s)$ lying in the upper half-plane to the real axis, and a_2 is the distance from the poles lying in the lower half-plane to the real axis. The class W_3 is found in the same way as the classes of correctness in the work of M. V. Fedoryuk ([1]).

If we take the intersection of the obtained classes of existence with the classes of uniqueness ([3,4,6]), then, since in our case the solution is the convolution of the initial data with the Green' s function, we obtain the classes of correctness.

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CITED LITERATURE

1. M. V. Fedoryuk, DAN, **132**, No. 1 (1960).

2. S. L. Sobolev, Izv. AN SSSR, ser. matem., **18**, No. 1, 3 (1954).

3. S. A. Galpern, DAN, **119**, No. 4 (1957).

4. S. A. Galpern, Tr. Mosk. matem. obshch., **9**, p. 401 (1960).

5. A. G. Kostyuchenko, G. I. Eskin, UMN, **15**, issue 2, 211 (1960).

6. I. M. Gel' fand, G. E. Shilov, *Some Questions in the Theory of Differential Equations*, Moscow, 1958.

* An analogous condition was formulated in works ([3,4]), and later in work ([5]).

Note: Figure translations are in progress. See original paper for figures.

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