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Abstract

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A. A. NOVRUZOV

ON PROPERTIES OF SOLUTIONS OF ELLIPTIC EQUATIONS

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In papers ([1−4]) a number of properties were obtained for solutions of elliptic equations of the form

$$\sum_{i,k=1}^{n} A_{ik}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} B_i(x)\frac{\partial u}{\partial x_i} + C(x)u = 0, \tag{1}$$

where $x = (x_1, x_2, \dots, x_n)$, which follow from the following lemma:

Let K_R be the ball of radius R with center at the origin O, and let $G \subset K_R$ be a domain containing the center of the ball and having limit points on the boundary of the ball. Let the volume of G be less than R^n/M, where M is a sufficiently large constant. Then, for a solution $u(x)$ of equation (1) which vanishes on that part of the boundary of the domain G which lies strictly inside K_R, the inequality

$$2u(0) \le \max_{x\in \overline{G}} u(x) \tag{2}$$

holds.

In these papers it is assumed that $C(x) \le 0$. The constant M in this case, generally speaking, increases as R increases. In those cases when an estimate independent of R is needed (in the proof of the Phragmén–Lindelöf theorem), it is assumed that $B_i(x) \equiv 0$.

The present note is devoted to inequality (2). In it an estimate uniform with respect to R is obtained when $B_i(x) \not\equiv 0$.

Denote by K_R the open ball in n-dimensional space of arbitrary radius R with center at the origin O, by $\mu_n E$ the n-dimensional Lebesgue measure of a set E lying in n-dimensional space. Denote by G a domain lying inside K_R, containing the center of the ball K_R and having limit points on the boundary of K_R, and by S^*_ρ the intersection of the domain G with the sphere of radius ρ and center at the origin O. Consider in G the equation

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0. \tag{3}$$

With respect to its coefficients it is assumed that in G the inequalities

$$\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k > \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \alpha > 0 \tag{4}$$

hold for any real ξ_i, $\sum_{i=1}^{n} \xi_i^2 \neq 0$,

$$(\operatorname{grad} a, \vec{\rho})\big|_G \geq 0, \qquad |a_{ik}(x)| \leq a_0; \tag{5}$$

$$\operatorname{div} \mathbf{b} \geq 0; \tag{6}$$

$$(\mathbf{b}, \vec{\rho})\big|_G \leq 0; \tag{7}$$

$$c(x) \leq 0, \tag{8}$$

where

$$a = \sum_{j=1}^{n} \sum_{i,k=1}^{n} \frac{\partial a_{ik}}{\partial x_j} \cos\alpha_i \cos\alpha_k, \qquad \mathbf{b} = (b_1, b_2, \dots, b_n),$$

and $\vec{\rho}$ is the radius vector from the center to the given point.

Lemma. Let Γ be that part of the boundary of the domain G which lies strictly inside K_R. Suppose that equation (3) is defined in G. There exists a constant M, depending only on the constants a and a_0 in inequalities (4), (5) and on the dimension n of the space, such that if, for every R,

$$\mu_n G < \mu_n K_R / M, \tag{9}$$

then, for any positive solution $u(x)$ of equation (3) in G, continuous in $\bar{G}$ and vanishing on Γ, the inequality

$$2u_0 < \max_{x \in \bar{G}} u(x), \qquad \text{where } u_0 = u(0).$$

holds.

Proof. Suppose the lemma is false. Denote by E_t the set of points $x \in G$ for which $u(x) = t,\ t \in [u_0/2, u_0]$. Let $E_t^*(r)$ be that part of E_t which lies inside K_r $(0 < r < R)$. Denote by G_t the set of points $x \in G$ at which $u(x) > t$. Let $G_r^t = G_t \cdot K_r$. Then, according to (3), there exist t_0 $(u_0/2 < t_0 < u_0)$ and a set $E \subset [0, R]$ of positive measure such that, for all $r \in E$,

$$\int_{E_{t_0}^*(r)} \left|\frac{\partial u}{\partial \tau}\right| dl > M u_0 \left(\mu_n G_r^t\right)^{(n-2)/n},$$

where τ is the exterior normal to the surface $E_{t_0}^*(r)$. Denote by G_1 the set of points $x \in G$ for which $u(x) > t_0$. Let $G_r = G_1 \cdot K_r,\ r \in [0, R]$. Consider the function $f(r) = \mu_n G_r$ on the interval $[0, R]$. By inequality (9), the function $f(r)$ satisfies all the conditions of Lemma 6 of the paper ([3]). Consequently, there exist nonintersecting intervals $[\alpha_1, \beta_1],\ [\alpha_2, \beta_2], \dots, [\alpha_s, \beta_s]$, belonging to the interval $[0, R]$, such that

$$\tfrac{1}{2} f(\alpha_j) \le f(\beta_j) \le 2 f(\alpha_j) \qquad (j = 1, 2, \dots, s);$$

$$f(r) \ge \min[f(\alpha_j), f(\beta_j)] \qquad \text{for } \alpha_j < r < \beta_j;$$

$$\beta_j - \alpha_j > \frac{R^{1/2n} M^{1/2n}}{C_n} f^{1/2n}(r); \qquad \sum_{j=1}^{s} (\beta_j - \alpha_j) > \frac{R}{4},$$

where C_n is a constant.

By the same lemma there exists such a $\lambda_j \in [\alpha_j, \beta_j]$, $j = 1, 2, \dots, s$, that

$$\int_{E_{t_0}^*(\lambda_j)} \left|\frac{\partial u}{\partial \tau}\right| dl > M u_0 \left(\mu_n G_{\lambda_j}\right)^{(n-2)/n}; \tag{10}$$

$$f(\beta_j) \le 2 f(\lambda_j); \tag{11}$$

$$\beta_j - \lambda_j > \frac{R^{1/2n} M^{1/2n}}{C_n} f^{1/2n}(r). \tag{12}$$

Further, by inequality (11), there exist r_0 $(\lambda_j < r_0 < (3\lambda_j + \beta_j)/4)$, r_1 $((\lambda_j + \beta_j)/2 < r_1 < \beta_j)$, such that

$$\mu_{n-1} S_{r_0}^* < \frac{\mu_n G_{\lambda_j}}{\beta_j - \lambda_j}; \tag{13}$$

$$\mu_{n-1} S^*_{r_1} < \frac{2\mu_n G_{\lambda_j}}{\beta_j - \lambda_j}. \tag{14}$$

Consider the integral

$$\int_{S^*_r} \sum_{i,k=1}^{n} a_{ik}(x) \cos\nu_i \frac{\partial u}{\partial x_k}\, ds,$$

where $\cos\nu_i$ are the direction cosines of the normal to S^*_r. In view of the inequalities (5), (13), and (14), there will be an r' $(r_0 < r' < r_1)$ such that

$$\int_{S^*_{r'}} \sum_{i,k=1}^{n} a_{ik}(x) \cos\nu_i \frac{\partial u}{\partial x_k}\, ds \le \frac{24\mu_0 a_0 n^2 \mu_n G_{\lambda_j}}{(\beta_j - \lambda_j)^2}. \tag{15}$$

Denote by $S_{r'}$ the boundary of the domain $G_{r'} = G_1 \cdot K_{r'}$. Clearly, $S_{r'}$ consists of points belonging to $E^*_{t_0}$ and $S^*_{r'}$. Applying Green' s formula in the domain $G_{r'}$, from equation (3) we obtain

$$\int_{E^*_{t_0}(r')} \sum_{i,k=1}^{n} a_{ik} \cos\nu_i \cos\nu_k \frac{\partial u}{\partial \tau}\, dl + \int_{S^*_{r'}} \sum_{k,i=1}^{n} a_{ik} \cos\nu_i \frac{\partial u}{\partial x_k}\, ds +$$

$$+ \int_{S^*_{r'}} (u - t_0) \sum_{i=1}^{n} b_i \cos\nu_i\, ds - \iint_{G_{r'}} (u - t_0) \operatorname{div} \mathbf{b}\, d\omega + \iint_{G_{r'}} cu\, d\omega = 0. \tag{16}$$

By virtue of inequalities (4) and (10),

$$\int_{E^*_{t_0}(r')} \sum_{k=1}^{n} a_{ik} \cos\nu_i \cos\nu_k \frac{\partial u}{\partial \tau}\, dl < -M\alpha u_0 \left(\mu_n G_{\lambda_j}\right)^{(n-2)/n}. \tag{17}$$

Further, by virtue of inequalities (6), (7), and (8) we have

$$\int_{S^*_{r'}} \sum b_i \cos\nu_i (u - t_0)\, ds \le 0, \qquad -\iint_{G_{r'}} (u - t_0) \operatorname{div} \mathbf{b}\, d\omega \le 0,$$

$$\iint_{G_{r'}} cu\, d\omega \le 0. \tag{18}$$

Combining inequalities (15), (17), and (18) with equality (16), we obtain

$$-M\alpha u_0(\mu_n G_{\lambda_j})^{(n-2)/n} + \frac{24a_0\mu_0 n^2\mu_n G_{\lambda_j}}{(\beta_j-\lambda_j)^2}C_n > 0,$$

whence, taking $M = 25n^2 a_0 C_n/\alpha$, by virtue of inequality (12), we have $u_0 < 0$, which is impossible. The lemma is proved. From this lemma the following theorems are obtained:

Theorem 1. Let the domain G lie inside the ball K_R (R arbitrary), contain the center of the ball K_R, and have limit points on the boundary of the ball K_R. Let in G there be defined a solution $u(x)$ of equation (3), vanishing on that part of the boundary of the domain G which is situated strictly inside K_R. Let the inequality $\mu_n G < \mu_n K_R/M$ hold, where by M here and everywhere below is denoted the constant of the lemma. Let $u_0 > 0$. Then

$$\sup u(x) > u_0 \exp(\mu_n K_R/M_1\mu_n G)^{\frac{1}{n-1}},$$

where M_1 is a positive constant depending only on the constants α, a_0 of inequalities (4), (5), and on the dimension n of the space.

Theorem 2. Let equation (3) be defined in the ball K_R. Let $u(x)$ be some solution of this equation in the ball K_R. Let $|u(x)| < 1$

in K_R^n. Let $G_1, G_2, \ldots$ be the maximal connected domains in which the function $u(x)$ preserves a constant sign. Let a $(0 < a < 1)$ be some number. Let $G_{i_1}, G_{i_2}, \ldots, G_{i_m}$ be those among the domains $G_1, G_2, \ldots$ which possess the following two properties:

1. The intersection $G_{i_k} \cap K_{R/2}$ is nonempty.
2. $\max\limits_{x\in G_{i_k}K_{R/2}} |u(x)| \geq a$.

Then the number m of these domains satisfies the inequality

$$m < M_2\left(\lg\frac{1}{a}\right)^{1/(n-1)},$$

where M_2 is a constant depending only on the constants α, a_0 in inequalities (4), (5), and on the dimension n of the space.

Theorem 3. Let G be an unbounded domain. Let equation (3) be given in G. Let R be an arbitrary positive number. Let σ be some other number satisfying the inequality

$$\sigma < \mu_n K_R/M. \tag{19}$$

Suppose the domain G has the property that for every ball K_R of radius R the inequality $\mu_n(G \cdot K_R) < \sigma$ holds. Let the domain G contain the origin O. Let there be defined in the domain G a solution $u(x) > 0$ of equation (3), which vanishes on the boundary of the domain. Put

$$M(r) = \max_{\sum_{i=1}^n x_i^2 = r^2} u(x).$$

Then, for all $r > R$, the inequality

$$M(r) > u_0 \exp \left(R^{n-1} r / M_3 \sigma\right)^{1/(n-1)}$$

holds, where M_3 is a positive constant depending only on the constants α, a_0 in inequalities (4), (5), and on the dimension n of the space.

Theorem 4 (Phragmén–Lindelöf). Let G be an unbounded domain having the following property. There exists a number $\eta < 1/M$ such that, if for a natural number m one denotes by G_m the intersection of G with K_{2^m} (K_{2^m} is the ball of radius 2^m with center at the origin O), then

$$-\mu_n G'_m / \mu_n K_{2^m} < \eta$$

for all m, beginning with some m_0. Let there be defined in G a solution $u(x)$ of equation (3), continuous in $\overline{G}$ and nonpositive on the boundary of the domain G. Then: 1) either $u(x) \leq 0$ everywhere in G; 2) or, if one puts

$$M(R) = \sup_{\sum_{i=1}^n x_i^2 = R^2} u(x),$$

then

$$\liminf_{R \to \infty} \frac{M(R)}{R^{(1/M_4 \eta)^{1/(n-1)}}} > 0,$$

where M_4 is a positive constant depending only on the constants α, a_0 in inequalities (4), (5), and on the dimension n of the space.

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Moscow State University
named after M. V. Lomonosov

Institute of Mathematics and Mechanics
Academy of Sciences of the Azerbaijan SSR

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Note: Figure translations are in progress. See original paper for figures.