SOME EXACT SOLUTIONS OF THE EQUATIONS OF UNSTEADY MOTIONS IN MAGNETOHYDRODYNAMICS

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Abstract

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SOME EXACT SOLUTIONS OF THE EQUATIONS OF UNSTEADY MOTIONS IN MAGNETOHYDRODYNAMICS

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In the present note the class of motions studied in works (^{1}-^{5}), in which the radial velocity is a linear function of the radius, is somewhat extended.

1. Pulsation of a gravitating sphere in a magnetic field. Unsteady flows of a gas with spherical symmetry, under the condition that the gas particles have radial motion with a velocity proportional to their radius, were investigated by L. I. Sedov (^{1}), and with the inclusion of gravitational forces by M. L. Lidov (^{2}). Here these solutions will be extended to the cases in which the gas, having infinite conductivity, is subjected to the action of an axially symmetric magnetic field, and also when the gas particles rotate relative to the axis of symmetry. The equations of magnetohydrodynamics in spherical coordinates under the condition of axial symmetry and $v_\theta = 0$ have the following form:

$$\begin{gathered}
\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = \frac{1}{4\pi\rho}\frac{H_\theta}{r}\left(\frac{\partial r H_\theta}{\partial r} - \frac{\partial H_r}{\partial \theta}\right) - \frac{H_\varphi}{4\pi\rho r}\frac{\partial}{\partial r}(rH_\varphi) - \frac{1}{\rho}\frac{\partial p}{\partial r} - f\frac{M}{r^2},\\
\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = \frac{H_r}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta\, H_\varphi) + \frac{H_\theta}{r^2\sin\theta}\frac{\partial(r\sin\theta\, H_\varphi)}{\partial\theta},\\
-\frac{v_\varphi^2}{r}\operatorname{ctg}\theta = -\frac{H_\varphi}{4\pi\rho r^2\sin\theta}\frac{\partial(r\sin\theta\, H_\varphi)}{\partial\theta} - \frac{H_r}{4\pi\rho r}\left(\frac{\partial H_\theta r}{\partial r} - \frac{\partial H_r}{\partial\theta}\right) - \frac{1}{\rho r}\frac{\partial p}{\partial\theta},\\
\frac{\partial}{\partial r}(H_r r^2\sin\theta) + \frac{\partial}{\partial\theta}(H_\theta r\sin\theta) = 0,\\
\frac{\partial\rho}{\partial t} + v_r\frac{\partial\rho}{\partial r} + \rho\left(\frac{2v_r}{r} + \frac{\partial v_r}{\partial r}\right) = 0,\\
\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \gamma p\left(\frac{\partial v_r}{\partial r} + \frac{2v_r}{r}\right) = 0,\\
\frac{\partial H_r}{\partial t} + v_r\frac{\partial H_r}{\partial r} + H_r\frac{2v_r}{r} = 0,\\
\frac{\partial H_\theta}{\partial t} + v_r\frac{\partial H_\theta}{\partial r} + H_\theta\left(\frac{v_r}{r} + \frac{\partial v_r}{\partial r}\right) = 0,\\
\frac{\partial H_\varphi}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(H_r v_\varphi - v_r H_\varphi)\right] + \frac{1}{r}\frac{\partial}{\partial\theta}(v_\varphi H_\theta).
\end{gathered}\tag{1}$$

Here $\gamma = c_p/c_v$ is the ratio of heat capacities; all the remaining quantities have the generally accepted notation;

$$M = 4\pi\int_0^r r^2\rho\, dr = 4\pi\int_0^\xi \eta^2\rho_0(\eta)\, d\eta,$$

ξ is the initial radius of the particle. We shall seek a solution in which $v_\varphi = 0$, $v_r = -r\mu'(t)/\mu(t)$, where $\mu(t)$ is an arbitrary function of time. In such a motion all particles undergo a homogeneous deformation $r = \xi/\mu(t)$.

It is easy to see that, with such a choice of the radial velocity, the last 5 equations of (1) become linear. Their general solutions have the form

$$\rho = \mu^{3\xi}\varphi(\xi), \quad p = \mu^{3\gamma}\Pi'_\xi(\xi,\theta), \quad H_r = \mu^2 h_r(\xi,\theta), \tag{2}$$

$$H_\theta = \mu^2 h_\theta(\xi,\theta), \quad H_\varphi = \mu^2 h_\varphi(\xi,\theta).$$

The first three equations (1) in this case can be solved if the pressure is represented as the sum of two terms

$$p = \mu^{3\gamma}[\Pi_1(\xi,\theta) + \Pi_2(\xi)].$$

The first of them is a function of two variables ξ, θ, while the second is a function only of the initial radius ξ. If the gradient of the first part of the pressure is balanced by the Lorentz force

$$\operatorname{grad} \mu^{3\gamma}\Pi_1(\xi,\theta) = \frac{1}{4\pi}[\mathbf{H}\times \operatorname{rot}\mathbf{H}], \tag{3}$$

then, for the second part of the pressure $\Pi_2(\xi)$ and the function $\mu(t)$, the equations (1) will hold:

$$\frac{d\Pi_2}{d\xi}\,\xi + 4\pi f\varphi(\xi)\int_0^{\xi}\eta^3\varphi(\eta)\,d\eta = \frac{N}{2}\xi^3\varphi(\xi); \tag{4}$$

$$\frac{d\mu}{dt} = \pm\mu^2(N\mu + L); \tag{5}$$

$\varphi(\xi)$ is an arbitrary function, and N and L are arbitrary constants. Equation (3) can be satisfied only for $\gamma = 4/3$. Representing $\mathbf{h}$ in the form (6)

$$\mathbf{h} = \mathbf{i}_z \times \vec{\xi}\,T(\xi,\theta) + \operatorname{rot}\,[\mathbf{i}_z \times \vec{\xi}\,P(\xi,\theta)],$$

we obtain from equations (1), (3) that the functions $P(\xi,\theta)$, $T(\xi,\theta)$ must satisfy the equations

$$\Delta_5 P + \frac{1}{\sigma^2}G(\sigma^2 P) = \Phi(\sigma^2 P); \tag{6}$$

$$\frac{d}{d(\sigma^2 P)}(\sigma^4 T^2) = 2G(\sigma^2 P). \tag{7}$$

Here $G(\sigma^2 P)$, $\Phi(\sigma^2 P)$ are arbitrary functions; $\Delta_5 P$ is the Laplace operator in the 5-dimensional space of initial states under the condition of axial symmetry; $\sigma = \xi\sin\theta$ is the distance from the axis of symmetry. Using, in particular, Prendergast' s solution (7) and assuming, for simplicity, the density to be a linear function of the radius ξ, one can construct a solution of the problem of the pulsation of a magnetic sphere of finite radius R. The characteristics of the motion will then have the expressions:

$$v_r = -r\frac{\mu'(t)}{\mu(t)}, \qquad \xi = r\mu(t), \qquad \rho = \mu^3(t)\rho_0\left(1 - \frac{\xi}{R}\right),$$

$$H_r = \frac{2K}{a^2}\mu^2(t)\cos\theta\left[1 - \left(\frac{R}{\xi}\right)^{1/2}\frac{J_{1/2}(a\xi)}{J_{3/2}(aR)}\right],$$

$$H_\theta = -\frac{K}{a^2}\mu^2(t)\sin\theta\left[2+\frac{(aR)^{1/2}}{J_{3/2}(aR)}\left\{\frac{J_{5/2}(a\xi)}{(a\xi)^{1/2}} - 2\frac{J_{3/2}(a\xi)}{(a\xi)^{1/2}}\right\}\right],$$

$$H_\psi = \frac{K}{a}\xi\mu^2(t)\sin\theta\left[1-\left(\frac{R}{\xi}\right)^{1/2}\frac{J_{3/2}(a\xi)}{J_{3/2}(aR)}\right], \tag{8}$$

$$p = \mu^4(t)\left\{-\frac{K^2}{4\pi a^2}\sin^2\theta\left[1-\left(\frac{R}{\xi}\right)^{1/2}\frac{J_{3/2}(a\xi)}{J_{3/2}(aR)}\right] + p_0\right.$$

$$\left.-\frac{\pi f\rho_0^2}{4R^2}\xi^4 + \left(\frac{2p_0}{R^3}+\frac{1}{2}\frac{\pi f\rho_0^2}{R}\right)\xi^3 - \left(\frac{3p_0}{R^2}+\frac{1}{4}\pi f\rho_0^2\right)\xi^2\right\},$$

$$\frac{d\mu}{dt} = \pm\mu^2(N\mu+L)^{1/2}.$$

Here $J_p(a\xi)$ is a Bessel function of the first kind; a satisfies the equation $J_{5/2}(aR) = 0$; K, p_0, ρ_0, L are arbitrary constants, which must be assigned so that the equalities

$$p > 0, \qquad \frac{N}{2} = \frac{6p_0}{R^2\rho_0} - \frac{5}{6}\pi f\rho_0 > 0. \tag{9}$$

hold.

If, in addition, the condition $L < 0$ is satisfied, then all gas particles execute periodic pulsational motions from the initial position toward the center of symmetry, followed by expansion to some maximum distance and a return to the initial position. It is easy to see that at the boundary of the sphere $\xi = R$ the pressure, density, and magnetic field vanish.

In the case when $\gamma = 5/3$ and the magnetic field is absent, one can construct a solution corresponding to radial pulsation of rotating gas particles. In this case

$$v_r = -r\frac{\mu'(t)}{\mu(t)}, \qquad \xi = r\mu, \qquad v_\varphi = A\mu(t)\sqrt{\xi}\sin\theta,$$

$$\rho = \mu^3(t)\rho_0, \qquad p = p_0 + A^2\rho_0\mu^5\xi\sin\theta + \frac{\rho_0 N}{2}\xi^2,$$

$$\frac{d\mu}{dt} = \mu^2\left(N\mu^2+K\mu+L\right)^{1/2}.$$

Here A, N, L, p_0, ρ_0 are arbitrary constants; $K = \frac{8}{3}\pi f\rho_0$. Under the condition $K^2 - 4LN > 0,\ N < 0$, all gas particles rotate with variable angular velocity, and their distances from the center of symmetry periodically decrease and increase. The smallest radius of a particle is $r_{\min} = \xi/\mu_2$, and the largest is $r_{\max} = \xi/\mu_1$; μ_1 and μ_2 are the smaller and larger positive roots of the equation $N\mu^2 + K\mu + L = 0$. During collapse the angular velocity of rotation of a particle increases; during expansion it decreases.

II. Pulsations of a rotating plasma cylinder. The equations of magnetic hydrodynamics of an infinitely conducting gas with cylindrical symmetry have the form

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{1}{8\pi\rho}\frac{\partial H_\varphi^2}{\partial r} - \frac{1}{4\pi\rho}\frac{H_\varphi^2}{r} - \frac{1}{8\pi\rho}\frac{\partial H_z^2}{\partial r} - \frac{2fM}{r},$$

$$\frac{\partial v_\varphi^2}{\partial t} + v_r \frac{\partial v_\varphi^2}{\partial r} + \frac{2}{r} v^2_{r v_\varphi} = 0,$$

$$\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \rho\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right) = 0, \tag{10}$$

$$\frac{\partial p}{\partial t} + v_r \frac{\partial p}{\partial r} + \gamma p\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right) = 0,$$

$$\frac{\partial H_\varphi^2}{\partial t} + v_r \frac{\partial H_\varphi^2}{\partial r} + 2H_\varphi^2 \frac{\partial v_r}{\partial r} = 0,$$

$$\frac{\partial H_z^2}{\partial t} + v_r \frac{\partial H_z^2}{\partial r} + 2H_z^2\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right) = 0.$$

Here

$$M = 2\pi \int_0^\xi \eta \rho_1(\eta)\, d\eta;$$

ξ is the initial radius; $\rho_1(\xi)$ is the initial density; f is the gravitational constant.

It is easy to see that the system is satisfied by the following particular solutions of the type considered here:

$$v_r = r\frac{\zeta'(t)}{\zeta(t)}, \qquad \rho = \zeta^{-2}\frac{\varphi'(\xi)}{\xi}, \qquad p = \zeta^{-2\gamma}F(\xi), \qquad H_\varphi^2 = \zeta^{-2}\xi F_1(\xi), \tag{11}$$

$$H_z^2 = \zeta^{-4}F_2(\xi), \qquad v_\varphi^2 = \zeta^{-2}\xi^2\Phi(\xi).$$

Here $\xi = r/\zeta(t)$ is the Lagrangian coordinate corresponding to the initial radius; $\varphi(\xi)$ and $\Phi(\xi)$ are arbitrary functions; $F(\xi), F_1(\xi), F_2(\xi)$ are related to the functions $\varphi(\xi)$ and $\Phi(\xi)$ by the relations

$$F(\xi) = A\varphi(\xi) + N,$$

$$\frac{1}{8\pi}\frac{d}{d\xi}\{\xi F_1(\xi)\} + \frac{1}{4\pi}F_1(\xi) + \frac{4\pi f}{\xi^2}\varphi(\xi)\varphi'(\xi) = B\varphi'(\xi), \tag{12}$$

$$\frac{1}{8\pi}\frac{dF_2(\xi)}{d\xi} - \varphi'(\xi)\Phi(\xi) = D\varphi'(\xi);$$

$\zeta(t)$ is a function of time satisfying the equation

$$\left(\frac{d\zeta}{dt}\right)^2 = \frac{A}{\gamma-1}\zeta^{-2(\gamma-1)} - 2B\ln\zeta + D\zeta^{-2} + C = f(\zeta). \tag{13}$$

Equation (13) was investigated by A. G. Kulikovskii (^{3}), I. M. Yavorskaya (^{4}), and E. V. Ryazanov (^{5}). Depending on the values of the arbitrary constants A, B, C, D, the gas particles may, while rotating about the axis, fly toward it, scatter to infinity, or have periodic pulsating radial motions.

The solution considered here depends, for arbitrary γ, on two arbitrary functions $\varphi(\xi)$ and $\Phi(\xi)$, which characterize the initial distribution of density and angular velocity. For $\gamma = 2$ the solution will depend on three arbitrary functions. In this case the initial distribution of pressure may be prescribed arbitrarily.

The solutions obtained may be applied to the study of radial pulsations of a rotating plasma cylinder of finite radius R. In this case, at the boundary of the cylinder, for $\xi = R$, the condition of equality of the total pressures outside and inside the cylinder must be satisfied. This can be achieved by means of an external magnetic field produced by a surface current. The current strength can be determined similarly to how this was done in (^{3}).

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