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Abstract

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MATHEMATICS

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NONNEGATIVE EIGENFUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR AND BROWNIAN MOTION IN SOME SYMMETRIC SPACES

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1. Our aim is to describe all nonnegative solutions of the differential equation

$$Af - cf = 0, \tag{1}$$

where c is a constant and A is the Laplace-Beltrami operator in a certain symmetric space E having negative curvature. We shall give a solution of this problem for the case when the motion group of the space E is isomorphic to the complex unimodular group of order l. An analogous solution can also be given for other complex semisimple Lie groups.

The basis of the present investigation is Martin' s method ([1]), used by him to describe all nonnegative solutions of the Laplace equation in an arbitrary domain of Euclidean space.

2. Let L be an l-dimensional complex Euclidean space; G the group of all linear transformations of the space L with determinant 1; E the set of all transformations $x \in G$ to which there corresponds a positive definite Hermitian form $(x\xi, \eta)$ $(\xi, \eta \in E)$. To each $g \in G$ there corresponds a transformation S_g of the space E, defined by the formula $S_g x = g^* x g$. The characteristic roots of any operator $x \in E$ are positive and may be written in the form $e^{\rho_1}, e^{\rho_2}, \dots, e^{\rho_l}$, where $\rho_1 \geq \rho_2 \geq \dots \geq \rho_l$ and $\rho_1 + \dots + \rho_l = 0$. We shall agree to denote the set $(\rho_1, \dots, \rho_l)$ by $\rho(x)$. In the space E there exists a Riemannian metric $d(x, y)$, invariant with respect to all transformations S_g $(g \in G)$. This metric is determined uniquely up to a constant factor. It is completely determined if one requires that

$$\frac{d(e,x)}{|\rho(x)|} \to 1$$

as $|\rho(x)| \to 0$.* We denote by A the Laplace–Beltrami operator corresponding to this metric.

Put $\delta = (\delta_1, \dots, \delta_l)$, where $\delta_j = \frac{1}{2}(l+1-2j)$. It is proved that for $c < -\delta^2$ every nonnegative solution of equation (1) is equal to zero. Therefore in what follows one may assume that $c \geq -\delta^2$.

3. A function $h(x,y)$ will be constructed satisfying the following conditions: a) for each $y \in E$, $h(x,y)$, as a function of x, satisfies equation (1) in the domain $E \setminus y$; b)

$$\lim_{y \to x} \frac{h(x,y)}{d(x,y)^{N-2}} = 1,$$

where $N = l^2 - 1$ is the dimension of the space E; c) $h(x,y) \to 0$ as $d(x,y) \to \infty$. We shall call such a function the Green' s function for equation (1).

Let $I_\nu(z)$ and $K_\nu(z)$ be Bessel functions of imaginary argument ([2], No. 3.7). For even l put

$$\Phi_l(z) = \pi^{1/2} 2^{2q-1/2} (2q)! \, [(4q)!]^{-1} z^{2q+1/2} \left(I_{-2q-1/2}(z) + I_{2q+1/2}(z) \right) =$$

$$= e^{-z} \sum_{k=0}^{2q} \frac{1}{k!} \binom{2q}{k} \binom{4q}{k}^{-1} (2z)^k,$$

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* By e is denoted the identity transformation. For any set of real numbers $\rho = (\rho_1, \dots, \rho_l)$ we put $\rho^2 = \rho_1^2 + \dots + \rho_l^2$, $|\rho| = (\rho^2)^{1/2}$.

where $q = \frac{1}{4} l^2 - 1$. For odd l put

$$\Phi_l(z) = [(\nu-1)!]^{-1} 2^{1-\nu} z^\nu K_\nu(z) = \sum_{m=0}^{\nu-1} (-1)^m (m!)^{-2} \binom{\nu-1}{m}^{-1} 2^{-2m} z^{2m} +$$

$$+ \sum_{m=0}^{\infty} [m!(\nu+m)!(\nu-1)!]^{-1} \left[2 \ln \frac{1}{2} z - \psi(m+1) - \psi(\nu+m+1) \right],$$

where $\nu = \frac{1}{2}(l^2 - 3)$, $\psi(m+1) = 1 + \frac{1}{2} + \dots + \frac{1}{m} - C$ (C is Euler' s constant).

Theorem 1. *For each $c \geq -\delta^2$, equation (1) has a Green' s function, which is given by the formula*

$$h(x,y)=\Phi_l(a|\rho|)|\rho|^{3-l^2}\prod_{j<k}\frac{\rho_j-\rho_k}{\operatorname{sh}^{1/2}(\rho_j-\rho_k)},$$

where $\rho=\rho(x^{-1/2}yx^{-1/2})$ *and* $a=(\delta^2+c)^{1/2}$. *This function is everywhere positive, and as* $d(x,y)\to\infty$

$$h(x,y)\sim a_l e^{-a|\rho|}|\rho|^{\frac{1}{2}(3-l^2)}\prod_{j<k}\frac{\rho_j-\rho_k}{\operatorname{sh}^{1/2}(\rho_j-\rho_k)},$$

where a_l *is a certain constant.*

4. We shall call a solution f of equation (1) **minimal** if $f\geq 0$ and if every nonnegative solution $\tilde{f}$ subject to the inequality $\tilde{f}\leq f$ differs from f only by a constant factor. Let B denote the set of all bases of the space L, and let R denote the set of all sets $\rho=(\rho_1,\dots,\rho_l)$ of real numbers satisfying the conditions $\rho_1\geq\dots\geq\rho_l$, $\rho_1+\dots+\rho_l=0$. For each $b=(e_1,\dots,e_l)\in B$, $\rho\in R$, put

$$f_{b,\rho}(x)=\prod_{k=1}^{l}[d_{b,k}(x)]^{-1-\rho_k+\rho_{k+1}},$$

where

$$d_{b,k}(x)=\begin{vmatrix}(xe_1,e_1) & \dots & (xe_1,e_k)\\ \dots & \dots & \dots\\ (xe_k,e_1) & \dots & (xe_k,e_k)\end{vmatrix},\qquad \rho_{l+1}=1-\delta_l.$$

Put $\rho\in R_c$ if $\rho\in R$ and $\rho^2=\delta^2+c$.

Theorem 2. *The set of minimal solutions of equation (1) coincides with the set of functions* $f_{b,\rho}(x)$ $(b\in B,\rho\in R_c)$.

The group G acts in the space B by the formula $g(e_1,\dots,e_l)=(ge_1,\dots,ge_l)$. It is not difficult to see that $f_{b,\rho}(gx)=f_{gb,\rho}(x)$.

Consider the totality of all orthonormal bases of the space E, and denote by V the set obtained from this totality by identifying proportional bases. (We call the bases $\{e_j\}$ and $\{e'_j\}$ proportional if $e'_j=\lambda_j e_j$, where λ_j are certain complex numbers.) Since the functions $d_{b,k}$ and $f_{b,\rho}$ are the same for all proportional bases b, it makes sense to speak of the functions $d_{v,k}$, $f_{v,\rho}$, corresponding to $v\in V$.

Theorem 3. *Every minimal solution of equation (1) can be represented, and moreover uniquely, in the form* $af_{v,\rho}$ $(a>0,\ v\in V,\ \rho\in R_c)$. *The formula*

$$f(x) = \int_{V \times R_c} f_{v,\rho}(x)\, d\mu$$

establishes a one-to-one correspondence between all nonnegative solutions of equation (1) and all finite measures* on $V \times R_c$.

5. As is known [3], to each symmetric polynomial Q in the l variables $\rho_1, \ldots, \rho_l$ there corresponds a certain differential operator $A(Q)$ in the space E, commuting with all shifts $f(x) \to f(S_g x)$ $(g \in G)$. (In particular, to the polynomial $Q = \rho_1^2 + \cdots + \rho_l^2 - \delta^2$ there corresponds the Laplace-Beltrami operator.) Functions that are eigenfunctions for all the operators $A(Q)$ are called spherical functions.

Theorem 4. The set of all nonnegative spherical functions is given by the formula

$$f(x) = \int_V f_{v,\rho}(x)\, d\mu, \tag{2}$$

where $\rho \in R$; μ is an arbitrary finite measure on V. The pair ρ, μ is uniquely determined by the function f. If f is given by formula (2), then for every Q

$$A(Q) f = Q(\rho) f.$$

The unitary operator g leaves invariant the set of all orthonormal bases of the space L, and therefore induces a certain transformation of the set V. Denote by μ_0 the probability measure on V invariant with respect to all such transformations.

A spherical function f is called zonal if $f(S_g x) = f(x)$ for all unitary $g \in G$. It follows from Theorem 4 that every nonnegative zonal spherical function can be written in the form

$$f(x) = \int_V f_{v,\rho}(x)\, d\mu_0$$

(cf. [6]); in particular,

$$1 = \int_V \pi(x, v)\, d\mu_0,$$

where

$$\pi(x, v) = f_{v,\delta}(x) = \prod_{k=1}^{l-1} d_{v,k}(x)^{-2}.$$

Theorem 5. For $c \neq 0$, all nonzero nonnegative solutions of equation (1) are unbounded. The set of all bounded solutions of the equation $Af = 0$ is given by the formula

$$f(x) = \int_V \pi(x, v) F(v)\, d\mu_0,$$

where F is an arbitrary bounded Borel function on V. The function F is determined by f uniquely up to a set of measure zero.

The main theorem of the paper [(4)] follows at once from Theorem 5.

6. For each operator $x \in E$ there exists an orthonormal eigenbasis. Denote by $v(x)$ the corresponding element of the space V. The function $v(x)$ is, generally speaking, multivalued, but if all characteristic roots of x are pairwise distinct, then $v(x)$ is uniquely determined by x.

To the differential operator A there corresponds a certain continuous Markov process x_t, which is called Brownian motion in the space E [(5)].

* All measures under consideration are assumed to be defined on the Borel subsets of the corresponding topological space.

Theorem 5. For any initial state x, almost surely the limits

$$\lim_{t\to\infty} \frac{\rho(x_t)}{|\rho(x_t)|} = \frac{\delta}{|\delta|}, \qquad \lim_{t\to\infty} v(x_t^*) = \eta.$$

exist. Here δ is the vector defined in item 2, and the probability distribution of η is given by the formula

$$P_x\{\eta \in \Gamma\} = \int_\Gamma \pi(x, v)\, d\mu_0.$$

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REFERENCES

1. R. S. Martin, Trans. Am. Math. Soc., **49**, 137 (1941).

2. G. N. Watson, *Theory of Bessel Functions*, vol. 1, Moscow, 1949.

3. F. A. Berezin, Tr. Mosc. Math. Soc., **6**, 371 (1957).

4. F. A. Berezin, DAN, **125**, No. 6, 1187 (1959).

5. K. Ito, Mem. College Sci. Univ. Kyoto, Ser. A, **28**, Mathematics, No. 1, 81 (1953).

6. I. M. Gel' fand, M. A. Naimark, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, **36** (1950).

Note: Figure translations are in progress. See original paper for figures.