



---

Soviet-era science, translated into English

# AVERAGING OF BALANCE EQUATIONS IN OCEANOLOGY

1961

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.39250>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**GEOPHYSICS**

**G. N. IVANOV-FRANTSEVICH**

**AVERAGING OF BALANCE EQUATIONS IN OCEANOLOGY**

*(Presented by Academician V. V. Shuleikin, 16 February 1961)*

The equations of motion, continuity, turbulent exchange of heat, salt, suspended matter, dissolved oxygen, etc., are balance equations for various quantities (momentum, mass, heat, salt, etc.).

If  $E$  is the amount of some quantity associated with a mass  $M$  filling a volume  $\tau$ , then

$$\delta E = e \delta M = e \rho \delta \tau, \tag{1}$$

where  $\rho$  is the mass density, and  $e$  is the amount of the quantity  $E$  per unit mass.

The balance equation of  $E$  has the form

$$\frac{\partial}{\partial t}(e\rho) + \frac{\partial}{\partial x_k} C_k[E] = \sigma[E], \tag{2}$$

where  $C_k$  is the component of the flux  $C$  of the quantity  $E$  along the axis  $x_k$ ;  $\sigma[E]$  is the intensity of sources and sinks of the quantity  $E$ . In summing from 1 to 3 we shall not write the summation sign over indices that enter twice into one and the same term.

The velocity  $c$  of the elements  $E$  is determined by the formula

$$c_k = \frac{C_k}{e\rho} \tag{3}$$

and may not coincide with the velocity  $v$  of the mass elements.

Then the balance equation takes the form

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_k}(\rho e c_k) = \sigma. \tag{4}$$

Taking

$$c_k = v_k + a_k,$$

where  $a_k$  is the deviation of the velocity of the elements  $E$  from the velocity of the mass elements to which  $E$  is referred, we decompose the flux into the sum of the advective flux  $\rho e v_k$  and the non-advective flux  $\rho e a_k$

$$\rho e c_k = \rho e v_k + \rho e a_k. \quad (5)$$

Different methods of averaging equation (4) lead to different equations of turbulent exchange. The role of the “reference” velocity field  $v_k$ , whose choice is determined only by convenience of calculation and by the formulation of the problem, is essential. We shall use time averages:

a) Reynolds:

$$\bar{e} = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} e dt,$$

b) weighted mean:

$$\hat{e} = \overline{\rho e} / \bar{\rho}.$$

Averaging the continuity equation according to Reynolds, we have

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{\rho} \bar{v}_k) + \frac{\partial}{\partial x_k} (\overline{\rho' v'_k}) = 0, \quad (6)$$

where  $\rho' = \rho - \bar{\rho}$ .

The presence of a nonadvective mass flux means that the velocity field  $\bar{v}_k$  does not determine the trajectories of the averaged mass  $\bar{\rho}$ . The latter are determined by the field  $\hat{v}_k$ . Indeed:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{\rho} \hat{v}_k) = 0, \quad (7)$$

i.e., relative to the field  $\hat{v}_k$  there is no nonadvective mass flux.

Assuming, without loss of generality,

$$\rho' v'_k = -B_{(k)} \frac{\partial \bar{\rho}}{\partial x_k} = -B_{\kappa j} \frac{\partial \bar{\rho}}{\partial x_j}, \quad (8)$$

where enclosing the index  $k$  in parentheses means that summation over  $k$  is not to be performed, we obtain the equation of “turbulent diffusion” of mass in two forms:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{\rho} \bar{v}_k) - \frac{\partial}{\partial x_k} \left( B_k \frac{\partial \bar{\rho}}{\partial x_k} \right) = 0 \quad (9)$$

or

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{\rho} \bar{v}_k) - \frac{\partial}{\partial x_k} \left( B_{kj} \frac{\partial \bar{\rho}}{\partial x_j} \right) = 0. \quad (9')$$

For an incompressible fluid, (9) passes into the system of two equations

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{v}_k \frac{\partial \bar{\rho}}{\partial x_k} - \frac{\partial}{\partial x_k} \left( B_k \frac{\partial \bar{\rho}}{\partial x_k} \right) = 0, \quad (10)$$

$$\frac{\partial \bar{v}_k}{\partial x_k} = 0. \quad (11)$$

Averaging the balance equation (4) in such a way that  $\bar{\rho}, \bar{\sigma}, \hat{e}, \hat{v}_k$  enter into it, we obtain

$$\frac{\partial}{\partial t} (\bar{\rho} \hat{e}) + \frac{\partial}{\partial x_k} (\bar{\rho} \hat{e} \hat{v}_k) + \frac{\partial}{\partial x_k} (\overline{\rho e'' v_k''} + \overline{\rho e \alpha_k}) = \bar{\sigma}, \quad (12)$$

where  $e'' = e - \hat{e}$ ,  $v_k'' = v_k - \hat{v}_k$ .

Assuming

$$\overline{\rho e'' v_k''} + \overline{\rho e \alpha_k} = -\hat{A}_{(k)} \frac{\partial \hat{e}}{\partial x_k} = -\hat{A}_{kj} \frac{\partial \hat{e}}{\partial x_j} \quad (13)$$

and using (7), we obtain the turbulent-diffusion equation in two forms

$$\bar{\rho} \left( \frac{\partial \hat{e}}{\partial t} + \hat{v}_k \frac{\partial \hat{e}}{\partial x_k} \right) - \frac{\partial}{\partial x_k} \left( \hat{A}_k \frac{\partial \hat{e}}{\partial x_k} \right) = \bar{\sigma} \quad (14)$$

or

$$\bar{\rho} \left( \frac{\partial \hat{e}}{\partial t} + \hat{v}_k \frac{\partial \hat{e}}{\partial x_k} \right) - \frac{\partial}{\partial x_k} \left( \hat{A}_{kj} \frac{\partial \hat{e}}{\partial x_j} \right) = \bar{\sigma}. \quad (14')$$

Equation (14') is Ertel's turbulent-diffusion equation (1). Ertel assumes the exchange tensor  $\hat{A}_{kj}$  to be symmetric. The proof of this is based on the postulate

that the nonadvective flux cannot be tangent to the isosurfaces of  $\hat{e}$ . This postulate is by no means obvious and requires further study.

If equation (4) is averaged in such a way that only Reynolds means enter, then, after neglecting some small terms,

one can arrive at the equations of turbulent diffusion

$$\bar{\rho} \left\{ \frac{d\bar{e}}{dt} + \bar{v}_k \frac{\partial \bar{e}}{\partial x_k} - \frac{\partial}{\partial x_k} \left( \bar{A}_k \frac{\partial \bar{e}}{\partial x_k} \right) \right\} - (\bar{A}_k + B) \frac{\partial \bar{\rho}}{\partial x_k} \frac{\partial \bar{e}}{\partial x_k} = \bar{\sigma} \quad (15)$$

or

$$\bar{\rho} \left\{ \frac{\partial \bar{e}}{\partial t} + \bar{v}_k \frac{\partial \bar{e}}{\partial x_k} - \frac{\partial}{\partial x_k} \left( \bar{A}_{kj} \frac{\partial \bar{e}}{\partial x_k} \right) \right\} - \left( A_{kj} \frac{\partial \bar{\rho}}{\partial x_k} \frac{\partial \bar{e}}{\partial x_j} + B_{kj} \frac{\partial \bar{e}}{\partial x_k} \frac{\partial \bar{\rho}}{\partial x_j} \right) = \bar{\sigma}. \quad (15')$$

Equation (15'), under the assumption that  $A_{kj} = B_{kj}$ , becomes Schmitz' s equation of turbulent diffusion (2)

$$\bar{\rho} \left\{ \frac{\partial \bar{e}}{\partial t} + \bar{v}_k \frac{\partial \bar{e}}{\partial x_k} - \frac{\partial}{\partial x_k} \left( A_{kj} \frac{\partial \bar{e}}{\partial x_k} \right) \right\} - A_{kj} \left\{ \frac{\partial \bar{\rho}}{\partial x_k} \frac{\partial \bar{e}}{\partial x_j} + \frac{\partial \bar{e}}{\partial x_k} \frac{\partial \bar{\rho}}{\partial x_j} \right\} = 0. \quad (16)$$

Equation (16) was obtained by Schmitz under the assumption that the fluid is incompressible. As we see, it has the same form also for a compressible fluid. For an incompressible fluid, in addition, (11) holds.

Thus, Ertel' s equation of turbulent diffusion (14') and the equivalent equation (14) hold for  $\hat{e}$  and for the "reference" velocity field  $\hat{v}_k$ . Schmitz' s equation (16) holds for  $\bar{e}$  and for the reference velocity field  $\bar{v}_k$ .

When weighted means are chosen for the velocity  $\hat{v}_i$  and the external forces  $\hat{X}_i$ , and Reynolds means for the remaining quantities, the averaged Reynolds equations of motion for variable density  $\bar{\rho}$  retain the same form as for constant density. If, however, Reynolds means are taken for all quantities entering the equations of motion, then the averaged equations of motion will have the form

$$\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_k \frac{\partial \bar{v}_i}{\partial x_k} + 2\Omega_{ki} \bar{v}_k = -\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} + \frac{\partial}{\partial x_k} \left( N_{(i)k} \frac{\partial \bar{v}_i}{\partial x_k} \right) + \frac{L}{\bar{\rho}}, \quad (17)$$

where  $N_{(i)k}$  is the kinematic coefficient of virtual viscosity,

$$L = \left( N_{(i)k} + B_k \right) \frac{\partial \bar{\rho}}{\partial x_k} \frac{\partial \bar{v}_i}{\partial x_k} + B_{(i)} \frac{\partial \bar{\rho}}{\partial x_i} \frac{\partial \bar{v}_k}{\partial x_k} + \frac{\partial}{\partial x_k} \left( \bar{v}_k B_{(i)} \frac{\partial \bar{\rho}}{\partial x_i} \right) - 2B_k \Omega_{ik} \frac{\partial \bar{\rho}}{\partial x_k}. \quad (18)$$

Recently in dynamical oceanology the “turbulent diffusion” equation for mass (10) has found application (3–5). Since the averaging system must be uniform, together with (10) one should use the equations of motion in the form (17). This circumstance, however, is not taken into account in any of the cited works, and the formulation of the problem in them is therefore insufficiently correct.

The “turbulent diffusion” of the quantity  $\bar{e}$  is a relative concept, closely connected with the choice of the “reference” velocity field. This choice is, in essence, arbitrary and is determined by the formulation of the problem and by the convenience of its mathematical treatment.

Thus, the balance equation of the averaged mass (9) includes a term with “diffusion” of mass relative to the field  $\bar{v}_k$ , whereas the balance equation (7) for the same averaged mass relative to the field  $\hat{v}_k$  has the usual form.

Institute of Oceanology  
Academy of Sciences of the USSR

Received  
16 II 1961

## CITED LITERATURE

1. H. Ertel, *Ann. d. Hydrographie*, **65**, No. 5, 193 (1937).
2. H. Schmitz, *Acta hydrophysica*, **2**, No. 4, 158 (1955).
3. K. Takano, *Bull. de l'Institut oceanographique*, No. 1128, 1, Monaco (1958).
4. W. Hansen, *Deutsche Hydrograph. Zs.*, **9**, No. 2, 102 (1956).
5. G. S. Lineykin, *Fundamental Questions of the Dynamical Theory of the Baroclinic Layer of the Sea*, L., 1957.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*