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Abstract

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SOME PROPERTIES OF AUTOMORPHISMS OF NILPOTENT GROUPS

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1. The main result of the note is the following theorem.

Theorem 1. *A finite set of automorphisms M of a nilpotent group G is nilpotent relative to G if and only if it is nilpotent relative to the factor group G/G' (here G' denotes the commutator subgroup of G).*

Some facts adjoining this theorem are given, and, in particular, a new proof of the following theorem of P. Hall (1):

If in a group G there is a nilpotent normal divisor N such that G/N' is a nilpotent group, then G is also a nilpotent group.

Let us recall the notation and definitions. Let G be a group and M some set of its automorphisms. Along with commutators formed from elements of the group G or automorphisms, we shall consider commutators of the form $[g, \sigma]$, where $g \in G$ and σ is an automorphism: $[g, \sigma] = g^{-1} \cdot \sigma(g)$. By $[A, M]$ $(A \subset G)$ we shall denote the subgroup generated by the commutators $[g, \sigma]$ for all $g \in A$, $\sigma \in M$. Further denote:

$$[G, M] = [G, M(1)], \dots, [G, M(n)] = [[G, M(n-1)], M];$$

$$[[\dots[[g, \sigma_{i_1}], \sigma_{i_2}], \dots], \sigma_{i_k}] = [g; \sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}].$$

A finite set M is called **nilpotent relative to the group** G if for every element $g \in G$ there is an exponent $n = n(g)$ such that, for $m > n$, all elements of the form $[g; \sigma_{i_1}, \dots, \sigma_{i_m}]$, where $\sigma_{i_k} \in M$, are equal to the identity of the group G. A group of automorphisms Γ of a group G is called **externally locally nilpotent** (locally nilpotent relative to G) if every finite subset of it is nilpotent relative to G. Γ is called **finitely stable** if in G there is a finite normal series of Γ-admissible subgroups, on all factors of which the elements of Γ induce identical automorphisms (such a series is called Γ-stable). Finally, the group Γ is called **locally finitely stable** if, for every subgroup F of Γ having a finite number of generators, in G there is a local system of F-admissible subgroups G_α, in each of which F acts as a finitely stable group. Here and below, when speaking of the action of a set of automorphisms of a group G with respect to subgroups and factor groups from G, we mean the action of the corresponding induced set of automorphisms.

It is known that the group of automorphisms Γ of a nilpotent group G is externally locally nilpotent if and only if it is locally finitely stable.

2. Everywhere below, $\mathfrak{G}$ is some group, Φ and Σ are its groups of automorphisms, with Σ invariant relative to Φ; H is a Φ- and Σ-admissible

normal divisor in $\mathfrak{G}$, and Σ induces the identity automorphisms in $\mathfrak{G}/H$ and H. By $\mathcal{E}$ and E we shall denote the identities, respectively, in the automorphism group and in the abstract group.

In note (2) the following assertion is given.

If $[\mathfrak{G}, \Phi] \subset H$, then for any $g \in \mathfrak{G}$ the relation $[g, [\Sigma, \Phi(n)]] = [[g, \Sigma], \Phi(n)]$ $(n = 1, 2, ...)$ holds. We shall show that in this relation the element g may be replaced by the whole group $\mathfrak{G}$, i.e., the following is true:

Lemma 1. *If* $[\mathfrak{G}, \Phi] \subset H$*, then for every* n *one has*

$$[\mathfrak{G}, [\Sigma, \Phi(n)]] = [[\mathfrak{G}, \Sigma], \Phi(n)].$$

Consider the case $n = 1$. The fact that $[\mathfrak{G}, [\Sigma, \Phi]] \subset [[\mathfrak{G}, \Sigma], \Phi]$ follows directly from (2), since the subgroup $[\mathfrak{G}, [\Sigma, \Phi]]$ is generated by the subgroups $[g, [\Sigma, \Phi]]$ over all $g \in \mathfrak{G}$.

To prove the reverse inclusion, first note that $[\mathfrak{G}, \Sigma]$, and in particular $[\mathfrak{G}, [\Sigma, \Phi]] = A$, is contained in the center of H. The generators in $[\mathfrak{G}, \Sigma]$ have the form $a_{ij} = [g_i, \sigma_j]$, where $g_i \in \mathfrak{G}$, $\sigma_j \in \Sigma$. For each such a_{ij} one has $[a_{ij}, \varphi] \in A$; here φ is an arbitrary element of Φ. Now let a and b be two elements of H such that $[a, \varphi] = x \in A$ and $[b, \varphi] = y \in A$. Then $\varphi(a) = ax$, $\varphi(b) = by$, and $\varphi(ab) = ax \cdot by = ab \cdot xy$. Moreover, $\varphi(a^{-1}) = a^{-1}x^{-1}$. Hence $[a^{\varepsilon_1} \cdot b^{\varepsilon_2}, \varphi] \in A$ for $\varepsilon_i = \pm 1$. Thus, if h is a product of positive and negative powers of elements of the type a_{ij}, then $[h, \varphi] \in A$, which proves the reverse inclusion: $[[\mathfrak{G}, \Sigma], \Phi] \subseteq [\mathfrak{G}, [\Sigma, \Phi]]$. For arbitrary n the formula is obtained by induction.

From the formula just given it follows, in particular, that if for some n $[\Sigma, \Phi(n)] = \mathcal{E}$, then $[[\mathfrak{G}, \Sigma], \Phi(n)] = E$, i.e., Φ is finitely stable relative to $[\mathfrak{G}, \Sigma]$.

The following lemma generalizes this fact to the case when H is a central subgroup in $\mathfrak{G}$.

Lemma 2. *Let* H *be a central subgroup in* $\mathfrak{G}$*, and let* Φ *induce a finitely stable group of automorphisms in* $\mathfrak{G}/H$*. Then, if for some* n

$$[\Sigma, \Phi(n)] = \mathcal{E},$$

then Φ *is finitely stable relative to* $[\mathfrak{G}, \Sigma]$.

Proof. Let the normal series

$$H = \mathfrak{G}_0 \subset \mathfrak{G}_1 \subset \dots \subset \mathfrak{G}_{i-1} \subset \mathfrak{G}_i \subset \dots \subset \mathfrak{G}_{n-1} \subset \mathfrak{G}_n = \mathfrak{G}$$

be a Φ-stable series. Since all $\mathfrak{G}_i$ contain H, they are all Σ-admissible. Put $H_i = [\mathfrak{G}_i, \Sigma]$. We obtain the series

$$E = H_0 \subset H_1 \subset \dots \subset H_{i-1} \subset H_i \subset \dots \subset H_{n-1} \subset H_n = [\mathfrak{G}, \Sigma],$$

all of whose terms are Φ-admissible normal divisors of the group $\mathfrak{G}$. We shall show that this series can be refined to a Φ-stable one.

Taking in Lemma 1 the subgroup $\mathfrak{G}_1$ in place of $\mathfrak{G}$, we see that Φ is finitely stable relative to H_1. We shall now prove that Φ is finitely stable relative to an arbitrary factor H_i/H_{i-1}. This will prove the lemma. Introduce the notation:

$$\overline{\mathfrak{G}} = \mathfrak{G}_i/H_{i-1}; \qquad \overline{H} = \mathfrak{G}_{i-1}/H_{i-1};$$

$\overline{\Sigma}$ and $\overline{\Phi}$ are the groups of automorphisms induced by the groups Σ and Φ in $\overline{\mathfrak{G}}$.

It is clear that $[\overline{\Sigma}, \overline{\Phi}(n)] = \overline{\mathcal{E}}$ and that $\overline{\Sigma}$ induces the identity in $\overline{H}$ and in $\overline{\mathfrak{G}}/\overline{H}$, while Φ does so in $\overline{\mathfrak{G}}/\overline{H}$. As before, we now obtain that $[[\overline{\mathfrak{G}}, \overline{\Sigma}], \overline{\Phi}(n)] = \overline{E}$, or

$$[[\mathfrak{G}_i, \Sigma], \Phi(n)] = [H_i, \Phi(n)] \subset H_{i-1}.$$

The lemma is proved.

The following theorem is contained, in implicit form, in the work of P. Hall ([1]).

Theorem 2. *Let G be a nilpotent group and Γ its group of automorphisms. Then, if Γ is finitely stable relative to G/G', then Γ is finitely stable also relative to the group G.*

Proof. We apply induction on the length s of the lower central series of the group G. For $s = 1$ the theorem is trivial. Suppose it already···

has been proved for all $s < k$, and let the series

$$G \supset G^{(1)} \supset G^{(2)} \supset \cdots \supset G^{(k-2)} \supset G^{(k-1)} \supset G^{(k)} = E \qquad (*)$$

be the lower central series of the group G of length k. In accordance with the induction hypothesis, we shall assume that Γ is finitely stable relative to $G/G^{(k-1)}$. Denote $G^{(k-2)} = \mathfrak{G}$, $G^{(k-1)} = H$; H is a central subgroup in $\mathfrak{G}$. Further, let Σ be the group of automorphisms induced by the inner automorphisms of the group G in $\mathfrak{G}$; let Φ be the group that Γ induces in $\mathfrak{G}$. It is clear that Σ induces the identity in H and $\mathfrak{G}/H$, that Φ is finitely stable in $\mathfrak{G}/H$, that $[\mathfrak{G}, \Sigma] = [\mathfrak{G}, G]$, and that Σ is invariant relative to Φ.

Since Γ is finitely stable relative to $G/G^{(k-1)}$, for some n $[G, \Gamma(n)]$ belongs to the center of the group G, and consequently $[\Sigma, \Phi(n)] = \mathcal{E}$. By Lemma 2 we immediately obtain that Φ is finitely stable in $[\mathfrak{G}, \Sigma] = G^{(k-1)}$, which proves the theorem.

From this theorem there follows immediately the theorem of P. Hall mentioned in the introduction.

3. Lemma 3. *Let H be a central subgroup in $\mathfrak{G}$. Suppose, further, that Φ has a finite number of generators and that in Σ there is a local system of subgroups Σ_α invariant relative to Φ, such that for each Σ_α there exists $n = n(\Sigma_\alpha)$ such that $[\Sigma_\alpha, \Phi(n)] = \mathcal{E}$. Then, if Φ is locally finitely stable relative to $\mathfrak{G}/H$, then Φ is locally finitely stable also relative to $[\Sigma, \mathfrak{G}]$.*

Proof. The subgroups of the form $[\mathfrak{G}, \Sigma_\alpha]$ constitute a local system in $[\mathfrak{G}, \Sigma]$, and all of them are Φ-admissible. Therefore it suffices to show that Φ is locally finitely stable relative to each of these subgroups. Since Φ has a finite number of generators, in $\mathfrak{G}/H$ there is a local system of Φ-admissible subgroups $\mathfrak{G}_\beta/H$, in each of which Φ induces a finitely stable group of automorphisms. Taking now in Lemma 2, as $\mathfrak{G}$, the subgroup $\mathfrak{G}_\beta$, and in place of Σ, Σ_α, we obtain that Φ induces a finitely stable group of automorphisms in $[\mathfrak{G}_\beta, \Sigma_\alpha]$. Since the subgroups $[\mathfrak{G}_\beta, \Sigma_\alpha]$, over all $\mathfrak{G}_\beta$, constitute a local system of $[\mathfrak{G}, \Sigma_\alpha]$, the assertion is proved.

We now prove Theorem 1. We shall assume that the theorem has been proved for all nilpotent groups of class $s < k$, and suppose that G is a nilpotent group whose lower central series $(*)$ has length k. Let M be a finite set of automorphisms of the group G, nilpotent relative to G/G'; M is also nilpotent relative to $G/G^{(k-1)}$. Denote by Γ the group of automorphisms of the group G generated by the set M. Since G is a nilpotent group, it follows from the nilpotency of M relative to $G/G^{(k-1)}$ that Γ is locally finitely stable relative to $G/G^{(k-1)}$. This means that in $G/G^{(k-1)}$ there is a local system of Γ-admissible subgroups $G_\alpha/G^{(k-1)}$, relative to each of which Γ acts as a finitely stable group—for each G_α there is an $n = n(G_\alpha)$ such that $[G_\alpha, \Gamma(n)]$ belongs to the center of the group G. In what follows we shall use the same notation as in the proof of Theorem 2. Denote also by Σ_α the subgroup in Σ corresponding to G_α. It is clear that Σ_α is invariant relative to Γ and, for $n = n(G_\alpha)$, one has $[\Sigma_\alpha, \Gamma(n)] = \mathcal{E}$, so that the hypotheses of Lemma 3 are satisfied; therefore Γ is locally finitely stable relative to $[\mathfrak{G}, \Sigma] = G^{(k-1)}$, which proves the theorem.

At the same time the following theorem has been proved:

Theorem 3. *Let G be a nilpotent group and Γ its group of automorphisms. Then Γ is locally nilpotent relative to G if and only if it is locally nilpotent relative to G/G'.*

From this theorem the following criterion for a locally nilpotent abstract group follows easily.

If in a group G there is a nilpotent normal divisor N such that G/N' is a locally nilpotent group, then G is also a locally nilpotent group.

Theorem 4. Let G be a nilpotent group, Γ its group of automorphisms, and Σ a normal divisor in Γ. Suppose further that Γ induces finitely stable (outer locally nilpotent) groups of automorphisms in $G/[G,\Sigma]$ and (through inner automorphisms) in Σ/Σ'. Then, if Σ is finitely stable relative to G, then Γ is also finitely stable (respectively, locally nilpotent) relative to G.

Proof. We consider only the case of outer local nilpotence. From the fact that Σ is finitely stable relative to G, it follows that Σ is a nilpotent group. But then, by the preceding theorem, Γ is locally nilpotent relative to Σ. Restricting ourselves to the case where Γ has a finite number of generators, we see that in Σ there is a local system of subgroups Σ_α invariant with respect to Γ, in each of which Γ induces a finitely stable group of automorphisms.

Let first G be an abelian group. Denote

$$[G,\Sigma] = G_1, \dots,\ [G_i,\Sigma] = G_{i+1}.$$

We obtain in G a descending series of Σ-commutants

$$G \supset G_1 \supset \cdots \supset G_{k-2} \supset G_{k-1} \supset G_k = E.$$

This series consists of Γ-admissible subgroups and is stable relative to Σ. If the series has length $k = 2$, then the theorem is contained directly in Lemma 3. Suppose it has already been proved for series of length $\leq k-1$. Then Γ is locally nilpotent relative to G/G_{k-1} and, in particular, relative to G_{k-2}/G_{k-1}. Applying Lemma 3 to the group G_{k-2}, we obtain that Γ is locally nilpotent relative to

$$[G_{k-2},\Sigma] = G_{k-1}.$$

Consequently, Γ is locally nilpotent relative to G.

Now consider the case where G is a nilpotent group. Since Σ is finitely stable relative to G, Σ is finitely stable also relative to the factor group G/G'. It is also clear that from the local nilpotence of Γ relative to $G/[G,\Sigma]$ there follows the local nilpotence of Γ relative to the factor group of the group G/G' by its Σ-commutant. Hence it follows that Γ is locally nilpotent relative to G/G'. By Theorem 3, Γ is locally nilpotent relative to G.

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[1] P. Hall, *Illinois J. Math.*, **2**, No. 4 B, 787 (1958). [2] B. I. Plotkin, V. G. Vil' yatser, DAN, **134**, No. 3, 529 (1960).

Note: Figure translations are in progress. See original paper for figures.