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Abstract

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MATHEMATICS

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PERTURBATION OF A LINEAR EQUATION BY A SMALL NONLINEAR TERM

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Let E_1 and E_2 be complex Banach spaces, and let λ be a small complex parameter. Consider the equation

$$By = f + \lambda \sum_{i=0}^{k} F_i y^i. \tag{1}$$

The unknown y is sought in E_1; f and F_0 belong to E_2; B is a normally solvable linear operator whose domain $D(B)$ is dense in E_1; the range $R(B)$ is closed in E_2, and the null spaces of the operators B and B^* are finite-dimensional. Decompose $D(B)$ into the direct sum $D(B) = E_1^n \dotplus \widetilde{D}(B)$, where E_1^n is the null subspace of the operator B, and $\widetilde{D}(B)$ is dense in $E_1^{\infty-n}$ (see ([1])). Let now $\widetilde{B}$ be the linear operator coinciding with B on $\widetilde{D}(B)$ and mapping $\widetilde{D}(B)$ onto $R_i(B)$ one-to-one. Put $\Gamma = \widetilde{B}^{-1}$. With regard to the operators $F_i y$, assume that these are i-linear operators whose domains of definition include $D(B)$, whose ranges lie in E_2, and which are bounded in the following sense: for $z \in R(B)$,

$$\left\| F_i \varphi^{i-s} (\Gamma z)^s \right\| \le C \|z\|^s, \qquad i \ge s; \quad i, s = 0, 1, \dots, k.$$

We note that the general scheme of equation (1) includes certain broad classes of nonlinear integral, singular integral, differential, and algebraic equations. The perturbation problem for nonlinear integral equations of the form (1) was studied in papers ([2–6]), and for matrices and differential equations in the linear case ($k = 1$), in paper ([7]). Introduce $\mathfrak{R}_k$ ($k \ge 2$), the class of abstract functions $y(\lambda)$, each defined and continuous in its own neighborhood of the point $\lambda = 0$, and having, as $\lambda \to 0$, order of growth

$$y(\lambda) = o\left(\lambda^{-\frac{1}{k-1}}\right).$$

The aim of the note is, first, to point out the elementary fact that all solutions of (1) in the class $\mathfrak{R}_k$ can be found with the aid of the general theory of branching; second, to give, in the simplest case, simple sufficient conditions for the existence of solutions of (1) in the class $\mathfrak{R}_k$. Here it turned out useful to transfer to the nonlinear case the notion of a generalized Jordan chain proposed in ([7]).

Put

$$g = y\mu, \qquad \mu = \lambda^{\frac{1}{k-1}}.$$

Now the problem of finding all solutions of equation (1) in the class $\mathfrak{R}_k$ is, obviously, equivalent to the problem of finding all small solutions of the equation

$$Bg = \mu f + \sum_{i=0}^{k} \mu^{k-i} F_i g^i. \tag{2}$$

Let us note that the results of paper [1] are also valid for unbounded operators. It suffices to use, instead of the Hildebrandt–Graves theorem, the analogous Theorem 2 of A. E. Gel' man [8]. The branching theory proposed in [1] makes it possible to find all small solutions of equation (2).

Everywhere below it is assumed that $n = m = 1$ (the notation of paper [1]).

Remark 1. From the Newton diagram method (as applied to the branching equation) it follows that all small solutions of equation (2) can be found in the form of convergent series in fractional powers of the parameter μ. But then, obviously, all solutions of (1) in the class $\mathfrak{N}_k$ are also representable by convergent series in fractional powers of λ, and only a finite number of terms can have negative exponents.

Introduce φ and ψ—null-elements, respectively, of the operators B and B^*.

Definition. The operator B has, relative to the operator $F_k y^k$, a Jordan chain of length p, if there exist p linearly independent elements $\varphi_1 = \varphi, \varphi_2, \dots, \varphi_p$, satisfying, for $s = 1, 2, \dots, p$, the relations

$$B\varphi_s = \sum_{s_1+s_2+\cdots+s_k=s+k-2} F_k \varphi_{s_1} \varphi_{s_2} \cdots \varphi_{s_k},$$

where

$$\psi \left(\sum_{s_1+s_2+\cdots+s_k=p+k-1} F_k \varphi_{s_1} \varphi_{s_2} \cdots \varphi_{s_k} \right) \neq 0.$$

Theorem 1. *Let the operator B have, relative to the operator $F_k y^k$, a Jordan chain of length p; then the number of single-valued complex solutions of equation (1) is equal to $1 + (k-1)p$.*

For the proof of this assertion, note that the length of the projection of the decreasing part of the Newton diagram of equation (2) is equal to $1 + (k-1)p$. A more detailed investigation makes it possible to establish more delicate facts.

Theorem 2. *Let $\psi(f) \neq 0$ and let the operator B have, relative to the operator $F_k y^k$, a Jordan chain of length p; then equation (1) has a special solution of the form*

$$y(\lambda) = \sum_{i=-p}^{\infty} y_i \lambda^{\frac{i}{1+(k-1)p}}. \tag{3}$$

For $k \geq 2$ this multivalued solution is unique in the class $\mathfrak{N}_k$. For $k = 1$ the solution is unique in the class of functions growing no faster than an arbitrary power.

Proof is obtained by substituting (3) into (1). The relation between the length of the Jordan chain and the solvability of the resulting recurrence system is used, as well as Remark 1. From Theorem 2, for $k = 1$, there follow certain results of M. I. Vishik and L. A. Lyusternik [7], and for $p = 1,\ k = 2$, Theorem 1 [4] of P. P. Rybin.

Let us proceed to the study of the more complicated case $\psi(f) = 0$. Here (1) may have, in $\mathfrak{N}_k$, both bounded and special solutions. Everywhere below, for simplicity, we put $k = 2$.

We begin with finding the bounded solutions of (1). We shall seek them in the form

$$y(\lambda) = \sum_{i=0}^{\infty} y_i \lambda^i. \tag{4}$$

We have the recurrence system

$$By_0 = f,$$

$$By_1 = A_0 + A_1 y_0 + A y_0^2,$$

$$\dots\dots\dots\dots$$

Since $y_0 = \Gamma f + c_0 \varphi$, the solvability condition for the second equation can be written in the form

$$\alpha c_0^2 + \beta c_0 + \gamma = 0, \tag{5}$$

where $\alpha = \psi(A_2\varphi^2)$; $\beta = \psi(2A_2\varphi\Gamma f + A_1\varphi)$; $\gamma = \psi(A_2(\Gamma f)^2 + A_1\Gamma f + A_0)$. If $\alpha \neq 0$, then, obviously, $p = 1$.

Theorem 3. If in (1) $k = 2,\ p = 1,\ \beta^2 - 4\alpha\gamma \neq 0$, then (1) has two bounded solutions of the form (4).

Theorem 4. If in (1) $k = 2,\ p = 1,\ \beta^2 - 4\alpha\gamma = 0$, then (1) has a bounded solution of the form

$$y(\lambda) = \sum_{i=0}^{\infty} y_i \lambda^{i/2}.$$

Theorem 5. If in (1) $k = 2,\ \beta \neq 0,\ B$ has, with respect to $F_k y^k$, a Jordan chain of length $p \geq 2$, then (1) has one bounded solution of the form (4) and a singular solution

$$y(\lambda) = \sum_{i=-p+1}^{\infty} y_i \lambda^{i/p}.$$

The proof of these theorems is obvious.

Let, finally, $\alpha = \beta = 0,\ \gamma \neq 0$. Equation (1) has no bounded solutions. The search for singular solutions presents considerable difficulties. For $p = 2$, (1) has one three-valued singular solution

$$y(\lambda) = \sum_{i=-1}^{\infty} y_i \lambda^{i/3}.$$

The case $p = 3$ was studied by V. V. Markman ([6]). The situation here becomes more complicated: (1) may have two two-valued singular solutions. The investigations for $p = 3$ have not been carried through to the end.

It seems to us inadvisable to continue studying new cases, since each concrete problem can be completely investigated by the Newton diagram method.

Remark 2. M. M. Smirnov ([4]) studied singular solutions of equation (1) of the form $y(\lambda) = O^*(\lambda^{-\frac{1}{k-1}})$. Let $y_0 \neq 0$ be a solution of the equation $By_0 = F_k y_0^k$. Put $y = y_0\lambda^{-\frac{1}{k-1}} + z$, where $z \in \mathfrak{N}_k$. To find z, the above exposition is applicable.

Remark 3. It seems plausible to the author that equation (1) has no solutions growing faster than $\lambda^{-\frac{1}{k-1}}$. This is confirmed by all known examples.

Theorem 6. Suppose that for all possible solutions of equation (1) an a priori estimate is known,

$$\|y(\lambda)\| \leq C\lambda^{-\alpha},$$

where $0 < \alpha < \frac{1}{k-1}$; then the Jordan chain B with respect to $F_k y^k$ is finite and its length

$$p \leq \frac{\alpha}{1 - \alpha(k-1)}.$$

and all solutions of (1) belong to the class $\mathfrak{N}_k$.

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