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## Abstract

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# On the Theory of Dynamical Systems with Multivalued and Discontinuous Characteristics

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## Mathematics

In applications one encounters dynamical systems of the form

$$\dot{x} = f(t, x), \tag{1}$$

about the right-hand sides of which we have incomplete information: for a given value of $t$ and a fixed $n$-dimensional vector $x$, the vector function $f(t, x)$ assumes not one value, but some set of values. The concept of systems with incomplete information is applicable, in particular, to so-called rough systems, usually considered as differential equations with discontinuous right-hand sides. Thus, for example, in the work ($^1$) equation (1) with a multivalued right-hand side was used to describe the motions of controlled systems containing elements with hysteresis and relay characteristics. The system (1) was then interpreted as a system of equations in contingencies ($^{2,3}$). Topological properties of the arrangement of trajectories of dynamical systems with nonuniqueness were studied by E. A. Barbashin ($^4$).

In the present note it is shown that in a number of cases the study of differential equations with multivalued right-hand sides can be reduced to the study of equations with single-valued right-hand sides, specified in a certain linear normed space. Thus, to systems with incomplete information, in particular to equations in contingencies and in paratingencies ($^{2,3}$), the well-developed theory of differential equations in linear spaces proves applicable.

1. Let $R$ be an $n$-dimensional Euclidean space, and $S$ a closed unit Euclidean ball. Let, further, $\gamma$ be some single-valued mapping of $S$ into $R$. The image of the ball $S$ in the space $R$ under the mapping $\gamma$ will be called an $S$-set. Consider the complete Banach space $M(R)$ of all measurable essentially bounded mappings $\gamma$ with norm

$$\|\gamma\| = \operatorname{vrai} \max_{p \subset S} \|\gamma(p)\|_R.$$

It is easy to see that the space $M(R)$ is isometric to the space of the corresponding $S$-sets $\gamma(S)$ of the space $R$ with metric $\|\gamma(S)\| = \operatorname{vrai\,max}_{p\subset S} \|\gamma(p)\|_R$. Therefore we shall sometimes identify an element $\gamma$ of the space $M(R)$ with the $S$-set $\gamma(S)$; however, we emphasize that we cannot construct the proposed theory by proceeding from the consideration only of certain subsets of the space $R$ without involving mappings $\gamma$, since definite difficulties arise in introducing a linear operation in the new space.

Let $E$ be an $m$-dimensional Euclidean space, and let $f(p)$ be a multivalued function defined on this space, whose values are certain $S$-sets of the space $R$. With the multivalued func-

we shall associate with this function $f(p)$ a single-valued function $F(p)$, whose values lie in $M(R)$ and are determined by the rule $F(p) = \gamma$, if and only if $\gamma(S) = f(p)$. This will allow us to transfer to multivalued functions all the notions of the descriptive theory of functions. Thus, for example, we shall call a multivalued function $f(p)$ **continuous** if the corresponding single-valued function $F(p)$ is continuous. Similarly we define the class of multivalued functions whose set of discontinuity points is a $G_\delta$ of the second category ($^5$): to these functions there correspond pointwise discontinuous functions $F(p)$ that are the limit of a sequence of continuous functions. The relay characteristic $f(x) = \operatorname{sign} x$, considered as multivalued (1) at the point $x = 0$, turns out to be precisely one of these functions (this function was also considered earlier ($^6$), p. 217) as the limit of a sequence of single-valued continuous functions).

2. To introduce the notion of the integral of a multivalued function, let us dwell on the notion of measurability of multivalued functions. We shall say that a sequence of multivalued functions $f_n(p)$ **converges almost uniformly** to the function $f(p)$ on the set $E' \subset E$, if for any $\varepsilon > 0$, $\delta > 0$ one can specify a set $E_\varepsilon \subset E'$, $\operatorname{mes} E_\varepsilon < \varepsilon$, and a positive integer $n(\delta)$ such that
$$\|F_n(p) - F(p)\|_{M(R)} < \delta$$
for all $p \subset E' \setminus E_\varepsilon$ and $n > n(\delta)$ (here $F_n(p)$ and $F(p)$ denote the single-valued functions with values in $M(R)$ corresponding to the multivalued functions $f_n(p)$ and $f(p)$). We shall call the function $f(p)$ **countably valued** on $E'$ if $F(E')$ is a countable set, and the inverse images of points of $M(R)$ under the mapping $F(p)$ of the space $E$ into $M(R)$ are measurable sets. Finally, we shall call a multivalued function $f(p)$ **measurable** if there exists a sequence of countably valued functions $f_n(p)$ converging to $f(p)$ almost uniformly on $E'$. Obviously, the corresponding single-valued function $F(p)$ will in this case be (strongly) measurable in the sense of Bochner ($^7$), p. 55).

We now define the integral of a measurable multivalued function $f(p)$ by the rule
$$\int_{E'} f(p)\, dp = (B) \int_{E'} F(p)\, dp,$$

where the integral on the right-hand side is the integral in the sense of Bochner ($^7$), p. 61).

Let $E$ be the number axis. We define the derivative of a multivalued function $f(t)$, $t \subset E$, by the rule

$$\frac{df(t)}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

where the limit is taken in the sense of the metric introduced above. According to ($^8$), we have:

$$\frac{d}{dt} \int_a^t f(t)\, dt = f(t).$$

3. Consider the differential equation (1), where $f(t,x)$ is a multivalued function defined for $x \subset R$, $-\infty < t < +\infty$ (the set $f(t,x)$ is an $S$-set of the space $R$). Suppose that the multivalued function $f(t,x)$ satisfies the following condition:

(A) If $X$ is an $S$-set of the space $R$, then $f(t,X)$ is also an $S$-set (here

$$f(t,X) = \bigcup_{x \subset X} f(t,x)$$

).

Along with equation (1), consider the differential equation

$$\frac{dX}{dt} = f(t,X), \tag{2}$$

solutions of which are regarded as multivalued functions $X(t)$ of the scalar argument $t$, with values that are $S$-sets of the space $R$ (the derivative in (2) is understood in the sense of the definition given above). The trajectories of system (2), in contrast to the trajectories of system (1), are specified at the initial instant not by points but, generally speaking, by certain $S$-sets of the space $R$, and are tubes situated in this space. The funnel trajectories of system (1) are included among the trajectories of system (2), since the points of the space $R$ are, obviously, $S$-sets.

If $X$ and $f(t,X)$ are regarded not as $S$-sets of the space $R$, but as elements of the space $M(R)$, then the differential equation (2) will be a differential equation with a single-valued right-hand side. Using the theory of differential equations in Banach spaces, it is not difficult to formulate, for equation (2), conditions for existence and also for continuability of solutions [9], to give definitions of the basic notions of stability theory [10,11], and so on. Under certain additional conditions, equation (2) will describe in $M(R)$ an ordinary dynamical system with the uniqueness property.

The proposed construction turns out to be more meaningful if, instead of the space $M(R)$, one considers the space $C(R)$ of **continuous** mappings $\gamma$. In this case, which apparently covers all systems (1) with multivalued right-hand sides encountered in applications, condition (A) will almost always be satisfied automatically.

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