MATHEMATICS

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1961

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Abstract

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REPRESENTATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR A SYSTEM OF PARABOLIC TYPE IN THE FORM OF A WIENER INTEGRAL

(Presented by Academician A. N. Kolmogorov on 18 II 1961)

In the present note we shall use the definitions and notation adopted in the paper (^{1}). By the Wiener integral of a matrix one should understand the matrix composed of the Wiener integrals of the corresponding elements of this matrix.

Consider the Cauchy problem for the system of differential equations*

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t)u(x,t) \tag{1}$$

with initial conditions

$$u(x,0) = \varphi(x); \tag{2}$$

where

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix}, \qquad \varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix},$$

$$p(x,t) = \begin{pmatrix} p_{11}(x,t), \dots, p_{1n}(x,t) \\ \cdots\cdots\cdots\cdots\cdots \\ p_{n1}(x,t), \dots, p_{nn}(x,t) \end{pmatrix}.$$

Theorem. Let $p_{ij}(x,t)$ $(i,j = 1,\dots,n)$ be continuous bounded functions in the domain $-\infty < x < +\infty,\ 0 < t < +\infty$; and let $\varphi_i(x)$ $(i = 1,\dots,n)$ be continuous and bounded functions for all real values of x. Then the solution of the Cauchy problem (1), (2) is representable in the form of the Wiener integral

$$u(x,t) = \int_C \exp\left\{ t\int_0^1 p\,[2\sqrt{t}\,y(\tau)+x,\ t(1-\tau)]\,d\tau \right\} \varphi[2\sqrt{t}\,y(1)+x]\,d_w y, \quad (3)$$

where the symbol exp denotes the matricant.

Proof. Finding the solution of the system of differential equations (1) under the initial conditions (2) is replaced by solving the following system of integral equations:

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi)^2}{4t}\right] \varphi(\xi)\,d\xi +$$

$$+\frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-t_1}} \left\{ \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi_1)^2}{4(t-t_1)}\right] p(\xi_1,t_1)u(\xi_1,t_1)\,d\xi_1 \right\} dt_1. \quad (4)$$

* After the present note had been prepared for publication, the paper (^{2}) was published.

Introduce the abbreviated notation

$$f(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi)^2}{4t}\right] \varphi(\xi)\,d\xi,$$

$$K(x,t;\xi_1,t_1) = \frac{1}{2\sqrt{\pi(t-t_1)}} \exp\left[-\frac{(x-\xi_1)^2}{4(t-t_1)}\right] p(\xi_1,t_1).$$

Now (4) takes the form

$$u(x,t) = f(x,t) + \int_0^t \int_{-\infty}^{+\infty} K(x,t;\xi_1,t_1)u(\xi_1,t_1)\,d\xi_1 dt_1. \quad (5)$$

Substituting in the right-hand side of system (5), in place of the column $u(\xi_1,t_1)$, its value delivered by the same system, we find

$$u(x,t) = f(x,t) + \int_0^t \int_{-\infty}^{+\infty} K(x,t;\xi_1,t_1)f(\xi_1,t_1)\,d\xi_1 dt_1 +$$

$$+\int_0^t \int_0^{t_1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K(x,t;\xi_1,t_1)K(\xi_1,t_1;\xi_2,t_2)u(\xi_2,t_2)\,d\xi_1 d\xi_2\,dt_1 dt_2.$$

Again substituting here, in place of $u(\xi_2, t_2)$, its value from (5), etc., leads to the following infinite series:

$$u(x,t) = \sum_{\nu=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{\nu-1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left[\prod_{j=1}^{\nu} K(\xi_{j-1}, t_{j-1}; \xi_j, t_j) \right] \times$$

$$\times f(\xi_\nu, t_\nu)\, d\xi_1 \cdots d\xi_\nu \, dt_1 \cdots dt_\nu,$$

where $t_0 \equiv t, \ \xi_0 \equiv x$.

Substitute into the series obtained, in place of K and f, their expressions, relabel the integration variable ξ as $\xi_{\nu+1}$, and simplify:

$$u(x,t) = \sum_{\nu=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{\nu-1}} \frac{dt_\nu \cdots dt_2 dt_1}{2^{\nu+1}\sqrt{\pi^{\nu+1}(t-t_1)(t_1-t_2)\cdots(t_{\nu-1}-t_\nu)t_\nu}} \times$$

$$\times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left[\prod_{j=1}^{\nu} p(\xi_j, t_j) \right] \varphi(\xi_{\nu+1}) \times$$

$$\times \exp\left[-\frac{(x-\xi_1)^2}{4(t-t_1)} - \frac{(\xi_1-\xi_2)^2}{4(t_1-t_2)} - \cdots - \frac{(\xi_{\nu-1}-\xi_\nu)^2}{4(t_{\nu-1}-t_\nu)} - \frac{(\xi_\nu-\xi_{\nu+1})^2}{t_\nu - t_{\nu+1}} \right] \times$$

$$\times d\xi_1 \cdots d\xi_{\nu+1}. \tag{6}$$

If we make the change of variables

$$t_j = t(1-\tau_j) \qquad (j = 1, \ldots, \nu),$$

$$\xi_j = 2\sqrt{t}\, y_j + x \qquad (j = 1, \ldots, \nu+1),$$

then (6) is transformed into

$$u(x,t) = \sum_{\nu=0}^{\infty} t^\nu \int_0^1 \int_{\tau_1}^1 \cdots \int_{\tau_{\nu-1}}^1 \frac{d\tau_\nu \cdots d\tau_2 d\tau_1}{\sqrt{\pi^{\nu+1}\tau_1(\tau_2-\tau_1)\cdots(\tau_\nu-\tau_{\nu-1})(1-\tau_\nu)}} \times$$

$$\times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left\{ \prod_{j=1}^{\nu} p\left[2\sqrt{t}\, y_j + x, \ t(1-\tau_j)\right] \right\} \varphi\left[2\sqrt{t}\, y_{\nu+1} + x\right] \times$$

$$\times \exp\left[-\frac{y_1^2}{\tau_1}-\frac{(y_2-y_1)^2}{\tau_2-\tau_1}-\cdots-\frac{(y_{\nu+1}-y_\nu)^2}{1-\tau_\nu}\right]dy_1\cdots dy_{\nu+1}, \tag{7}$$

and from this it is already easy to obtain equality (3).

The series (7) converges absolutely and uniformly. Indeed, it decomposes into scalar series, each of which is majorized by the convergent series

$$\sum_{\nu=0}^{\infty}\frac{t^\nu M_1^\nu M_2}{\nu!},$$

$$M_1=\max\left[|p_{11}(x,t)|,\ |p_{12}(x,t)|,\dots,\ |p_{nn}(x,t)|\right];$$

$$M_2=\max\left[|\varphi_1(x)|,\dots,\ |\varphi_n(x)|\right].$$

Let us consider a somewhat more general problem

$$\frac{\partial u(x,t)}{\partial t}=\Delta u(x,t)+p(x,t)u(x,t); \tag{1'}$$

$$u(x,0)=\varphi(x), \tag{2'}$$

where $x=(x_1,x_2,\dots,x_m)$.

Repeating the preceding arguments, it is not difficult to calculate that

$$u(x_1,\dots,x_m,t)=$$

$$=\int_{c^m}\exp\left\{t\int_0^1 p\left[2\sqrt{t}\,y_1(\tau)+x_1,\dots,2\sqrt{t}\,y_m(\tau)+x_m,\ t(1-\tau)\right]d\tau\right\}\times$$

$$\times\varphi\left[2\sqrt{t}\,y_1(1)+x_1,\dots,2\sqrt{t}\,y_m(1)+x_m\right]\,d_wy_1\cdots d_wy_m. \tag{3'}$$

The integral on the right-hand side of formula $(3')$ is a multiple Wiener integral (1).

If in equalities (3) and $(3')$ we put $n=1$, then we obtain the formulas found earlier by Cameron (3) and Tingley (4).

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Received
15 II 1961

CITED LITERATURE

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[2] Yu. L. Daletskii, *DAN*, **134**, No. 5, 1013 (1960).
[3] R. H. Cameron, *Ann. Math.*, **59**, No. 3, 434 (1954).
[4] A. J. Tingley, *Proc. Am. Math. Soc.*, **7**, No. 5, 846 (1956).

Note: Figure translations are in progress. See original paper for figures.

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