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Abstract

Full Text

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CONTINUAL INTEGRALS AND CHARACTERISTICS ASSOCIATED WITH A GROUP OF OPERATORS

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1. Let $\mathfrak{H}$ be a Hilbert space, T a self-adjoint positive-definite operator whose domain D, with norm $\|\varphi\|_+ = \|T\varphi\|$, forms a space of basic elements; N the space conjugate to D—the space of generalized elements with norm $\|\xi\|_- = \|T^{-1}\xi\|$ (^{1}). By $(\varphi, \xi) = (\xi, \varphi)$ is denoted the value of the functional $\xi \in N$ on the basic element φ, coinciding, for $\xi \in \mathfrak{H}$, with the scalar product in $\mathfrak{H}$.

Let $\mathfrak{B}$ be some space with measure $\sigma(x)$. Suppose that in N there exists a complete system of elements $\{\xi_x\}(x \in \mathfrak{B})$, i.e., a system satisfying, for any $\varphi, \psi \in D$, the condition

$$(\varphi, \psi) = \int_{\mathfrak{B}} (\varphi, \xi_x)(\xi_x, \psi)\, d\sigma(x).$$

Below, almost always $\mathfrak{B}$ is the space of pairs of the form $x = (a, k)$, where a is an element of Euclidean space E_r and $1 \le k \le \nu$. In this case integration over $\mathfrak{B}$ also includes summation over the index k, but for convenience we shall not explicitly write the sums, understanding the corresponding expressions as vector- or matrix-valued with respect to the indices not indicated.

To each basic element $\varphi \in D$ we associate by definition the basic function $\varphi(x) = (\varphi, \xi_x)$ on $\mathfrak{B}$, and to each generalized element $\eta \in N$ the generalized function $\eta(x) = (\eta, \xi_x)$, defined by the relation

$$\int_{\mathfrak{B}} (\varphi, \xi_x)(\xi_x, \eta)\, d\sigma(x) = (\varphi, \eta).$$

We shall assume that multiplication by sufficiently smooth bounded functions is a continuous operation in D. If V is an operator whose adjoint acts in D, then we extend it to N, and the generalized kernel $(V\xi_x, \xi_y)$ is meaningful. For sufficiently smooth bounded functions $a_1(x), \dots, a_{n-1}(x)$ and operators $V_1, \dots, V_n$ of the type described, the generalized function

$$\int_{\mathfrak{B}} \cdots \int_{\mathfrak{B}} a_1(x_1) a_2(x_2) \cdots a_{n-1}(x_{n-1}) (\xi_{x_0}, V_1 \xi_{x_1}) \cdots (\xi_{x_{n-1}}, V_n \xi_x) \prod_{k=1}^{n-1} d\sigma(x_k),$$

is meaningful, where the integral converges in the weak sense.

2. Let A be a self-adjoint operator with domain $D_A = D$. By analogy with ([2],[3]), we define the continual integral of a functional $\Phi[x(\tau)]$ over the space $M(x_0, x)$ of bounded functions $x(\tau)$ on $[0, t]$ with values in $\mathfrak{B}$, satisfying the conditions $x(0) = x_0$, $x(t) = x$, as the limit $I(\Phi) = \lim_q I_q(\Phi)$ over all possible partitions

$$q\ (0 < t_1 < \dots < t_{n-1} < t)$$

of the expression

$$I_q(\Phi) = \int_{\mathfrak{B}} \cdots \int_{\mathfrak{B}} \Phi[x_q(\tau)] \left\{ \prod_{k=1}^{n} \left(\xi_{x_{k-1}}, e^{-iA(t_k - t_{k-1})} \xi_{x_k} \right) \right\} d\sigma(x_1) \cdots d\sigma(x_{n-1}).$$

Here $x_q(\tau) = x_k = x(t_k)$ for $t_k \leqslant \tau < t_{k+1}$ $(k = 0, \dots, n-1)$. If this expression has meaning, then, generally speaking, it is a generalized function, and the limit is understood in the weak sense. We shall denote

$$I(\Phi) = \int_{M(x_0,x)}^{*} \Phi[x(\tau)]\, d\mu_{iA}[x(\tau)].$$

If $x = (\alpha, k)$, $y = (\beta, i)$, then an expression of the form $(\xi_x, V\xi_y)$ is a matrix, which, omitting indices, we shall denote by $(\xi_\alpha, V\xi_\beta)$. Suppose moreover that $I_q(\Phi)$ reduces to the form

$$I_q(\Phi) = \int_{E^r} \cdots \int_{E^r} (\xi_{\alpha_0}, e^{-iA\Delta t_1} \xi_{\alpha_1})[1 + \gamma(\alpha_1)\Delta t_1](\xi_{\alpha_1}, e^{-iA\Delta t_2} \xi_{\alpha_2}) \cdots$$

$$\cdots [1 + \gamma(\alpha_{n-1})\Delta t_{n-1}](\xi_{\alpha_{n-1}}, e^{-iA\Delta t_n} \xi_\alpha) \prod_{k=1}^{n-1} d\sigma(\alpha_k),$$

where $\gamma(\alpha_k)$ are certain matrices. The continual integral obtained after passage to the limit will be denoted by the symbol

$$(T) \int_{M(\alpha_0,\alpha_1)} \exp\left\{ \int_0^t \gamma(\alpha(\tau))\, d\tau \right\} d\mu_{iA}[\alpha(\tau)]. \tag{1}$$

The sign T denotes, as is customary in quantum field theory, that the noncommuting factors following it must be arranged in order of increasing τ.

3. If C is some operator in $\mathfrak{H}$, then under certain conditions (^{2,3}) the formula

$$e^{(iA+C)t}\varphi = \lim_{q} \prod_{k=1}^{n} e^{iA\Delta t_k} e^{C\Delta t_k}\varphi \quad (\varphi \in D),$$

is valid in the sense of strong convergence in D, whence there follows, for the generalized kernel

$$W(x_0, x, t) = (e^{(iA+C)t}\xi_{x_0}, \xi_x),$$

the representation

$$W(x_0, x, t) = \lim_{q} \int_{\mathfrak{B}} \cdots \int_{\mathfrak{B}} \left\{ \prod_{k=1}^{n} \left(e^{iA\Delta t_k} e^{C\Delta t_k} \xi_{x_{k-1}}, \xi_{x_k} \right) \right\} d\sigma(x_1) \cdots d\sigma(x_{n-1}), \tag{2}$$

which makes it possible to express, in the form of continual integrals, the fundamental solutions of the operator equation

$$\partial\psi/\partial t = (iA + C)\psi.$$

In particular, one obtains formulas for differential equations or systems of Schrödinger type, analogously to how this was done in (^{2,3}) for parabolic ones; for example, the following theorem holds:

Theorem 1. *The fundamental matrix of the system*

$$\partial\psi/\partial t = iL(\psi) + C(x,t)\psi,$$

where $L(\psi)$ is a strongly elliptic system of differential operators, and $C(x,t)$ is a bounded smooth matrix-function, is representable in the form

$$W(a_0, a_1 t) = (T) \int_{M(a_0,a_1)}^{*} \exp\left\{ \int_0^t C[a(\tau), \tau]\, d\tau \right\} d\mu_{iL}$$

in the sense of convergence of generalized functions with respect to the variable ${x_0}^*$.

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[*] In the interesting work of R. Cameron (^{5}), continual integrals connected with the Schrödinger equation were constructed from functionals not necessarily of the form

$$\exp \int_0^t \Phi[x(\tau)]\, d\tau,$$

but analytic ones.

Representations for systems of equations of the form

$$\frac{\partial \psi}{\partial t} = iL(\psi) + \sum_{j=1}^{v} a_j \frac{\partial \psi}{\partial x_j} + C\psi,$$

are obtained by a somewhat more complicated route; we do not give them because of lack of space. For example, for $L = \Delta$ one obtains a formula similar to formula (11) of (^{3}), but with the imaginary coefficient $a = i$. (It was derived from heuristic considerations in (^{4}).) Such equations have already been considered by us in (^{3}), but there, for their solutions, only representations were obtained in the form of a limit of continual integrals corresponding to the equations $\partial\psi/\partial t = (i + \varepsilon)L(\psi) + C\psi$ as $\varepsilon \to 0$. We note that the consideration can be carried out both in the whole space and in a domain with certain correct boundary conditions. The symbol T before the integral may be omitted if one of the matrices L or C is a scalar multiple of the identity.

All these results generalize to the case of operators depending on t.

4. We have considered the application of formula (2) to the representation of solutions of equations of Schrödinger type. This formula shows (see (^{4,6})) how the fundamental solution is formed from contributions corresponding to individual trajectories. In the general case, in formula (2), before passage to the limit, for a fixed partition q, all step functions corresponding to this partition participate, and therefore a nonzero contribution to the result is given at least by trajectories from some everywhere dense set. We shall now consider the case corresponding to hyperbolic systems, when only trajectories having characteristic directions give a nonzero contribution to the result. In doing so, an abstract description of this case will be given.

Let B be a self-adjoint operator with finite-multiple spectrum; $E(\Delta)$ the spectral family corresponding to it; $P_1, \dots, P_\nu$ the operators projecting onto the invariant subspaces of the operator B, in which its spectrum is simple. Suppose that, for a dense set of functions $f(\alpha) \in \mathcal{L}_2$, the domains of definition D_A and $D_{f(B)}$ of the operators A and $f(B)$ have a common part D'_f, dense in $\mathfrak{G}$, such that $e^{iAt}D'_f \subset D_{f(B)}$, $f(B)D'_f \subset D_A$, $P_k D'_f \subset D'_f$, and there exist real functions $S_k(t, x)$ satisfying, for $\varphi \in D'_f$, the condition

$$\varepsilon_k(f, t)\varphi = \{e^{iAt} f(B) P_k - f[S_k(t, B)] e^{iAt} P_k\}\varphi = o(t). \tag{3}$$

We shall then say that the functions $S_k(t, x)$ determine the characteristics of the group of operators e^{iAt} in the representation associated with the operator B.

For $v = 1$, formula (3) implies the existence of a group of shifts $S(t, x)$ on the real axis satisfying the condition

$$e^{iAt} f(B) = f(S(t, B)) e^{iAt}. \tag{4}$$

We note that formula (4) makes it possible to solve the Cauchy problem for the equation $d\psi/dt = iA\psi$ with an arbitrary initial vector, if its solution is known when the initial vector is taken to be a generating element for the operator B.

Theorem 2. *In order that the group of operators* e^{iAt} *possess characteristics in the representation associated with the operator* B*, it is necessary and sufficient that*

$$AP_{kB} - BAP_k = -ig_k(B)P_k. \tag{5}$$

In this case $g_k(x) = S'_{kt}(0, x)$.

The conditions of the theorem make it possible to apply it to a hyperbolic system of differential equations of the form

$$\frac{\partial\psi}{\partial t} = a(x)\frac{\partial\psi}{\partial x} + b(x), \tag{6}$$

where a is a Hermitian matrix and b is an arbitrary matrix. It can be shown that, in fact, these equations exhaust the suitable examples.

More precisely, there exists an isomorphic mapping of $\mathfrak{H}$ onto some $\mathcal{L}_2(\sigma)$, under which the equation $d\psi/dt = iA\psi$, satisfying conditions (5), is transformed into an equation of type (6).

Analogous considerations can be developed for equations of the form

$$\frac{\partial\psi}{\partial t} = \sum_{j=1}^{r} \alpha_j \frac{\partial\psi}{\partial x_j} + \beta$$

with commuting matrices α_j.

Theorem 3. *The fundamental matrix of the equation* $\partial\psi/\partial t = iA\psi$ *in the representation in which, for* ξ_x*, generalized eigen-elements of the operator* B *connected with* A *by relations (5) are taken, is representable in the form of a continual integral*

$$W(a_0, a, t) = (T)\int^{*}_{M(a_0,a)} \exp\left\{\int_0^t C[\alpha(\tau)]\right\} d\mu_{iA_1}[\alpha(\tau)], \tag{7}$$

where

$$A_1 = \sum_{j=1}^{\nu} P_j A P_j; \qquad C(x) \text{ is the matrix determined by the relation}$$

$$C(\alpha)(\xi_\alpha, \xi_\alpha) = \left(\sum_{j \neq k} P_j A P_k \xi_\alpha, \xi_\alpha \right).$$

In this case only trajectories tangent to the characteristics make a contribution different from zero to the result.

Let us note that the integral in (7) is, in the present case, a limit of sums, and not of integrals. In essence, this formula gives an abstract notation for the difference method of solving the equation $d\psi/dt = iA\psi$, analogous to the well-known method of characteristics.

Formula (7) is especially simplified and reduces to finite multiple integrals in the triangular case, when $P_i A P_k = 0$ for $i > k$. In this case a nonzero contribution is made only by curves that do not pass from characteristics with a larger number to characteristics with a smaller number.

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Note: Figure translations are in progress. See original paper for figures.