BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH POWER NONLINEARITIES

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Abstract

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BOUNDARY-VALUE PROBLEMS FOR QUASI-LINEAR ELLIPTIC EQUATIONS WITH POWER NONLINEARITIES

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1. In this note the basic boundary-value problems are studied for quasilinear elliptic equations of the second order with coefficients of power growth. The method used by us makes it possible to prove existence and uniqueness theorems for generalized solutions of the first, second, and third boundary-value problems, and also to obtain estimates for them, proceeding from the corresponding results for equations with bounded nonlinearities; the method does not depend on the type of boundary condition. The results for the case of the second and third boundary-value problems are new. The first boundary-value problem, from various points of view, has been studied by many authors (see ([1]), where a detailed bibliography is given ([4])).

2. Let Ω be a bounded domain in m-dimensional Euclidean space with twice continuously differentiable boundary Γ. For the equation

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i}(a_i(x_j, u, p_i)) + b(x_i, u, p_i) = f(x_i) \tag{1}$$

$$(p_i = \partial u/\partial x_i,\ a_i(x_j, u, 0) = b(x_i, 0, 0) \equiv 0)$$

the boundary-value problems are posed:

$$u\,\big|_{\Gamma} = 0; \tag{2}$$

$$\sum_i a_i \cos(\nu, x_i)\,\big|_{\Gamma} = 0, \qquad \sum_i a_i \cos(\nu, x_i) + \sigma u\,\big|_{\Gamma} = 0, \qquad \sigma \geqslant 0. \tag{3}$$

When $b(x_i, u, p_i) \equiv 0,\ \partial a_i/\partial u \equiv 0$, for solvability of the second boundary-value problem one has to make the usual assumptions.

We shall give a detailed exposition for equations of the particular form

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i}(a_i(p_i)) = f(x_i), \qquad f(x_i) \in \mathcal{L}_2(\Omega), \tag{4}$$

under the boundary conditions (2) or (3).

Equation (4) is assumed to be elliptic, with ellipticity coefficient determined by the order of growth of the coefficients a_i; the a_i are continuously differentiable functions of their arguments, satisfying the conditions

$$\sum_{i=1}^{m} a_i p_i \geqslant \gamma T^n \qquad (T = |\operatorname{grad} u|), \qquad n \geqslant 2;$$

$$\sum_{i,j=1}^{m} \frac{\partial a_i}{\partial p_j} \xi_i \xi_j \geqslant \gamma_1 (T^{n-2} + 1) \sum_{i=1}^{m} \xi_i^2;$$

$$\left| \frac{\partial a_i}{\partial p_i} \right| \leqslant C_1 (T^{n-2} + 1).$$

Generalized solutions of the boundary-value problems are understood in the sense of satisfying an integral identity.

The proposed method is as follows: a numerical parameter α, $0 < \alpha < \alpha_0$, is introduced into equation (4):

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(\frac{a_i}{1 + \alpha Q} \right) \equiv -\sum_{i,j=1}^{m} \frac{\dfrac{\partial a_i}{\partial p_j} + \alpha \left(\dfrac{\partial a_i}{\partial p_j} Q - a_i \dfrac{\partial Q}{\partial p_j} \right)}{(1 + \alpha Q)^2} u_{ij} = f(x_i), \tag{5}$$

where $Q = Q(p_i)$ is an auxiliary function chosen so that equation (5) is elliptic with an ellipticity coefficient independent of p_j and α, and, for fixed α, contains only bounded nonlinearities.

Existence theorems for generalized solutions of boundary-value problems for equations with bounded nonlinearities were proved by the author ([2]).

Equation (5) will be elliptic with bounded nonlinearities if the function Q satisfies the conditions:

$$\gamma_2 T^{n-2} \leq Q \leq C_2 T^{n-2}, \qquad \left| \frac{\partial Q}{\partial p_j} \right| \leq C_3 T^{n-3}; \tag{6}$$

$$\sum_{i,j=1}^{m} \left(\frac{\partial a_i}{\partial p_j} Q - a_i \frac{\partial Q}{\partial p_j} \right) \xi_i \xi_j \geq (\gamma_3 T^{2n-4} - C_4) \sum_{i=1}^{m} \xi_i^2. \tag{7}$$

Let us examine how the limiting passage with respect to the parameter α can be carried out as $\alpha \to 0$.

Theorem 1. *Generalized solutions u of the first or second boundary-value problem for equation (5), for any fixed α, $0 < \alpha < \alpha_0$, possess generalized second derivatives and satisfy equation (5) almost everywhere. For them the estimates*

$$\int_\Omega \frac{1+T^{n-2}}{1+\alpha T^{n-2}} \left(T^2 + \sum_{i,j=1}^{m} u_{ij}^2 \right) d\Omega \le C_5 \|f\|_{L_2}^2. \tag{8}$$

hold.

Inequality (8), both for the first and for the second boundary-value problem, is obtained by the device indicated in the work of O. A. Ladyzhenskaya ([3]).

Theorem 2. *Let $\{\alpha_k\}$ be a decreasing sequence of numbers $0 < \alpha_k < \alpha_0$, tending to zero. Then the corresponding sequence of generalized solutions converges in the norm W_2^1 to the generalized solution u of the original problem. For u the estimate*

$$\int_\Omega (1+T^{n-2}) \left(T^2 + \sum_{i,j=1}^{m} u_{ij}^2 \right) d\Omega \le C_6 \|f\|_{L_2}^2$$

holds.

3. The main difficulty in using the scheme set forth above consists in constructing the function Q. We shall assume that in the coefficients of equation (4) the homogeneous principal part A_i, relative to p_j, can be separated out.* Then as Q one may take the function

$$Q = \left(\sum_{i=1}^{m} A_{ip} i \right)^{\frac{n-2}{n}}. \tag{9}$$

That a function Q of the form (9) satisfies inequality (7) for $n \ge 3$ follows from the identity.**

* To this condition one must add the conditions of Theorem 1 in the author' s note ([4]).
** For $2 < n < 3$ one must use another identity, which we do not give.

$$\sum_{i,j=1}^{m} \left(A_{ij} Q - A_i \frac{\partial Q}{\partial p_j} \right) Q^{\frac{2}{n-2}} \xi_i \xi_j \equiv \frac{2}{n(n-1)} \sum_{i=1}^{m} A_i p_i \sum_{i,j=1}^{m} A_{ij} \xi_i \xi_j +$$

$$+\frac{(n+1)(n-2)}{4n(n-1)^2}\sum_{\beta,\delta} M_{\beta\delta}(p\xi)_\beta (p\xi)_\delta + \frac{n-2}{n(n-1)^2}\left(\sum_{i,j=1}^{m} A_{ij}p_i\xi_j\right)^2 +$$

$$+\frac{(n-3)(n-2)}{4n(n-2)^2}\left[\sum_{\beta}(A_{ij}-A_{ji})(p\xi)_\beta\right]^2, \qquad (10)$$

where $A_{ij} = \partial A_i/\partial p_j$, $\{\beta\}$ is the set of combinations of the numbers $1, 2, \dots, m$ taken two at a time; $M_{\beta\delta}$ are the second-order minors of the matrix $\|A_{ij} + A_{ji}\|_{i,j=1,2,\dots,m}$, formed so that β determines the combination with the row numbers, and δ with the column numbers. If $\beta = (i, j)$, then $(p\xi)_\beta = p_i\xi_j - p_j\xi_i$.

The summation in the second term on the right-hand side of identity (10) is carried out over all combinations. The quadratic form

$$\sum_{\beta,\delta} M_{\beta\delta}(p\xi)_\beta (p\xi)_\delta$$

with respect to the variables $(p\xi)_\beta$ is positive (see ([5])).

4. Suppose now that the equation has the form (1) and that its lower-order terms are, in a certain sense, subordinate to the higher-order ones. We give one variant of such subordination.

Represent $b(x_i, u, p_i)$ in the form of the sum

$$b(x_i, u, p_i) = b_1(x_i, u, p_i) + b_2(x_i, p_i),$$

where

$$b_1(x_i, 0, p_i) = b_2(x_i, 0) \equiv 0.$$

Let the coefficients of equation (1), a_i and b, satisfy conditions 1–4 of Theorem 1 and condition 6 of Theorem 2 of article ([4]), and the inequalities

$$\frac{\partial b_1}{\partial u} \le C_7\left(T^{n-2} + |u|^{n-2} + 1\right), \qquad b_1 u \ge K\left(T^{n-2} + |u|^{n-2} + 1\right)u^2.$$

Under these assumptions the solvability of the boundary-value problems under consideration can be obtained by the same method as for equation (4).

The parameter is introduced into equation (1) in the following way:

$$-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{a_i}{1+\alpha Q}\right) + \frac{b_1(x_i, u, p_i)}{1+\alpha\frac{b_1}{u}} + \frac{b_2(x_i, p_i)}{1+\alpha T^{n-2}} = f(x_i),$$

where

$$Q = \left(\sum_{i=1}^{m} A_i p_i \right)^{\frac{n-2}{n}} + d|u|^{n-2};$$

A_i is the principal part of a_i, homogeneous with respect to p_j; d = const, determined by the data of the problem.

Note added in proof. After the article had been submitted for publication, papers by M. V. Vishik ([6],[7]) appeared, in which analogous problems for systems are considered by another method.

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Note: Figure translations are in progress. See original paper for figures.