REDUCTION OF PERIODIC PROBLEMS OF MATHEMATICAL PHYSICS TO SINGULAR EQUATIONS WITH A CAUCHY KERNEL

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Abstract

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REDUCTION OF PERIODIC PROBLEMS OF MATHEMATICAL PHYSICS TO SINGULAR EQUATIONS WITH A CAUCHY KERNEL

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In the note ([1]) a class of problems of mathematical physics was indicated which, by means of the Fourier integral, reduce to the Riemann boundary-value problem in the theory of analytic functions. Below we consider a related class of periodic problems which, by means of the finite Fourier transform (Fourier series)

$$V\varphi \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)e^{-ikx}\,dx = \Phi_k, \qquad V^{-1}\Phi_k \equiv \sum_{k=-\infty}^{\infty} \Phi_k e^{ikx} = \varphi(x) \quad (1)$$

$$k = 0, \pm 1, \pm 2, \dots, \qquad -\pi < x < \pi,$$

reduce to integro-differential equations containing a singular Cauchy integral. The proposed method of reduction is illustrated on problems for the biharmonic equation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0. \quad (2)$$

§ 1. Let equation (2) be given in the plane with cuts lying on the axis $y = 0$ and repeating with period 2π. Using the formula

$$V\frac{\partial^{p+q} u(x,y)}{\partial x^p \partial y^q} = (ik)^p \frac{d^q U_k(y)}{dy^q}, \quad (3)$$

we transform equation (2) by Fourier in the half-planes $y < 0$ and $y > 0$, and write the solutions of the resulting ordinary differential equations:

$$U_k(y) = \begin{cases} A_{k1}e^{-|k|y} + yB_{k1}e^{-|k|y} + C_{k1}e^{|k|y} + yD_{k1}e^{|k|y}, & k \neq 0,\ y < 0, \\ y^2A_{01} + y^3B_{01} + C_{01} + yD_{01}, & k = 0,\ y < 0, \\ A_{k2}e^{-|k|y} + yB_{k2}e^{-|k|y} + C_{k2}e^{|k|y} + yD_{k2}e^{|k|y}, & k \neq 0,\ y > 0, \\ A_{02} + yB_{02} + y^2C_{02} + y^3D_{02}, & k = 0,\ y > 0. \end{cases} \quad (4)$$

We restrict the growth of the solution as $|y| \to \infty$, putting $A_{k1} = B_{k1} = C_{k2} = D_{k2} = 0$. To determine the remaining functions A_{k2}, B_{k2}, C_{k1}, and D_{k1}, we prescribe on the interval $y = 0$, $|x| < \pi$, the corresponding number of boundary conditions:

$$\sum_{p=0}^{3}\sum_{q=0}^{3}\left[\alpha_{jpqr}\frac{\partial^{p+q}u(x,+0)}{\partial x^p \partial y^q} + \beta_{jpqr}\frac{\partial^{p+q}u(x,-0)}{\partial x^p \partial y^q}\right] = f_r(x), \tag{5}$$

$$x_j < x < x_{j+1}; \qquad x_0 = -\pi, \quad x_{m+1} = \pi; \qquad j = 0, \dots, m \ (m \geq 0);$$

$$r = 1, 2, 3, 4.$$

These conditions are extended with period 2π to the entire line $y = 0$. If the segment $x_j < x < x_{j+1}$ of the line $y = 0$ does not belong to the boundary, then we set $\alpha_{j0q,q+1} = -\beta_{j0q,q+1} = 1$, $q = 0, 1, 2, 3$, and set the remaining coefficients and the right-hand side $f_r(x)$ equal to zero for $x_j < x < x_{j+1}$.

In order to apply the transformation (1) to the conditions (5), we define each of them on the entire interval $|x| \leq \pi$, adding to the right-hand sides functions $\varphi_{jr}(x)$ equal to zero for $x_j < x < x_{j+1}$ and unknown on the remaining part of this interval.

As a result of the transformation we arrive at the problem: find the functions A_{k2}, B_{k2}, C_{k1}, D_{k1}, and Φ_{kjr} of the discrete argument k, satisfying the conditions

$$\sum_{p,q=0}^{3}\left\{\alpha_{jpqr}(ik)^p\left[(-|k|)^q A_{k2} + q(-|k|)^{q-1}B_{k2}\right] + \beta_{jpqr}(ik)^p\left[|k|^q C_{k1} + q|k|^{q-1}D_{k1}\right]\right\} = F_{kr} + \Phi_{kjr}, \tag{6}$$

$$r = 1, 2, 3, 4; \qquad j = 0, \dots, m; \qquad k = 0, \pm 1, \pm 2, \dots,$$

where the inverse transforms of the functions Φ_{kjr} must have the property

$$\varphi_{jr}(x) = 0, \qquad x_j < x < x_{j+1}. \tag{7}$$

Performing the inverse transformation and using formulas (3) and

$$V^{-1}\Phi_k \operatorname{sgn}\left(k + \frac{1}{2}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\varphi(t)e^{it}\,dt}{e^{it} - e^{ix}}, \tag{8}$$

we reduce the "discrete" problem (6), (7) to the system of integro-differential equations:

$$\sum_{p,q=0}^{3}\left\{\alpha_{jpqr}\left[\frac{d^{p+q}}{dx^{p+q}}\left(a_2(x)\cos\frac{\pi q}{2}+\frac{i\sin\pi q/2}{\pi}\int_{-\pi}^{\pi}\frac{a_2(t)e^{it}\,dt}{e^{it}-e^{ix}}\right)+q\frac{d^{p+q-1}}{dx^{p+q-1}}\left(b_2(x)\sin\frac{\pi q}{2}-\frac{i\cos\pi q/2}{\pi}\int_{-\pi}^{\pi}\frac{b_2(t)}{e^{it}}\right.\right.\right.$$

$$+\beta_{jpqr}\left[\frac{d^{p+q}}{dx^{p+q}}\left(c_1(x)\cos\frac{\pi q}{2}-\frac{i\sin\pi q/2}{\pi}\int_{-\pi}^{\pi}\frac{c_1(t)e^{it}\,dt}{e^{it}-e^{ix}}\right)+q\frac{d^{p+q-1}}{dx^{p+q-1}}\left(d_1(x)\sin\frac{\pi q}{2}+\frac{i\cos\pi q/2}{\pi}\int_{-\pi}^{\pi}\frac{d_1(t)e^{it}\,d}{e^{it}-e^{ix}}\right.\right. \tag{9}$$

$$x_j < x < x_{j+1}, \quad j = 0, \dots, m; \qquad r = 1, 2, 3, 4.$$

Let us note that instead of a plane with cuts one may take, as the domain, a half-plane, for example $y > 0$. In the conditions (5) one then sets $r = 1, 2$ and $\beta_{jpqr} = 0$.

§ 2. The system (9), in a number of cases, is reduced to special integral equations with Cauchy kernel for an open contour, solvable in quadratures (see, for example, ([2]), §§ 107, 108). In this way one can solve, for example, the following problems.

1. **Periodic fundamental mixed problem for the half-plane** $y < 0$**.** Let G be a collection of a finite number of segments belonging to $[-\pi, \pi]$ and having no common points. On G the components of the displacement vector are prescribed; on $[-\pi, \pi] - G$ the components of the stress $\sigma_y(x, -0)$ and $\tau_{xy}(x, -0)$ are prescribed.
2. **Periodic contact problem in the presence of friction forces.** On G the value of the normal displacement and the relation $\tau_{xy}(x, -0) + \rho\sigma_y(x, -0) = 0$ are prescribed; on $[-\pi, \pi] - G$ the quantities $\sigma_y(x, -0)$ and $\tau_{xy}(x, -0)$ are prescribed.

Another method of solving the periodic contact problem without friction ($\rho = 0$) is set out in ([3]).

3. **The problem for the strip** $y > 0$, $0 < x < \pi$. Conditions: $u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0$; $u(x, 0) = 0$, $0 < x < \pi$; $u_{yy}(x, 0) = 0$, $x \in G$; $u_y(x, 0) = f(x)$, $x \in [0, \pi] - G$, $G \subset [0, \pi]$. This problem is reduced to one of the form (5) by an odd periodic continuation.

§ 3. Let now equation (2) be given in the plane with cuts that are periodically repeating segments of the straight lines $y = y_s$, $s = 1, \dots, n$; $-\infty < y_1 < \dots < y_n < \infty$. Instead of (5) we pose the problem

$$\sum_{p,q=0}^{3}\left[\alpha_{ipqrs}\frac{\partial^{p+q}u(x,y_s+0)}{\partial x^p\partial y^q}+\beta_{ipqrs}\frac{\partial^{p+q}u(x,y_s-0)}{\partial x^p\partial y^q}\right]=f_{rs}(x), \tag{10}$$

$$x_j<x<x_{j+1},\quad j=0,\dots,m;\qquad r=1,2,3,4;\qquad s=1,\dots,n.$$

By a method analogous to that described above, this problem is reduced to a discrete problem, and then to a system of integro-differential equations. In doing this, along with formulas (3) and (8), the convolution formula is used

$$V^{-1}A_k\Phi_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}a(x-t)\varphi(t)\,dt, \tag{11}$$

where $a(x)$ has period 2π.

Along with problem (10) for n straight lines, consider n problems for one straight line, where the boundary conditions of the s-th problem have the form (10), with s fixed. Let $K_s\varphi_s=g_s$ be a shorthand notation for the system of integro-differential equations (9) to which the s-th problem is reduced. Here φ_s is the unknown, and g_s is the given vector-function. Then the system of integro-differential equations for problem (10) for n straight lines can be represented in the form

$$K_s\psi_s+\sum_{i=1}^{n}T_{si}\psi_i=h_s,\qquad s=1,\dots,n, \tag{12}$$

where φ_s is the unknown, h_s are given vector-functions, and the elements of the matrix T_{si} are integrals of the form (11) with kernels $a(x-t)$, whose Fourier coefficients decrease exponentially as $|k|\to\infty$.

Thus, if there is a method for solving the systems $K_s\varphi_s=g_s$, then the system of integro-differential equations (12) can be solved approximately by replacing the functions $a(x)$ with segments of their Fourier series. This method of approximate solution is convenient both because of the rapid convergence of the series just mentioned and because the coefficients of these series are available explicitly in the discrete problem.

In addition to what was said at the end of § 1, we note that instead of a plane with cuts one may with equal success take as the domain a strip with cuts or a half-plane with cuts.

§ 4. Along with the problems for equation (2), problems for other linear equations with constant coefficients or coefficients depending on y (the equations $\Delta u+\lambda u=f$, $\Delta^n u=f$, etc., in Cartesian and polar coordinates), and for systems of such equations (for example, the system of equations of the theory of elasticity), can be reduced by the method indicated above to integro-differential equations and solved.

Instead of conditions (10), one may take more general ones, containing linear combinations of limiting values of derivatives of the unknown functions on different straight lines $y = y_s$ (example: a problem periodic also in y).

The number of independent variables may be greater than two. Obviously, those multidimensional problems reduce to systems of integro-differential equations which, after the application of integral transformations, reduce to two-dimensional problems of the class indicated above. For the equation $\Delta u = f$, such problems will be, for example, the following.

1. In Cartesian coordinates the domain is defined by the inequalities $-\pi < x < \pi$, $-a < y < a$, $-b < z < b$. Conditions: $u(-\pi, y, z) = u(\pi, y, z) = 0$; $u(x, -a, z) = u(x, a, z) = 0$; $u(x, y, -b) = 0$; $u(x, y, b) = g(x, y)$ for $x \in G$; $u_z(x, y, b) = h(x, y)$ for $x \in [-\pi, \pi] - G$.
2. In cylindrical coordinates, where x is the polar angle, the domain is given by the inequalities: $0 < z < l$, $0 < \lambda < r < 1$, $|x| < \pi\alpha$, $0 < \alpha \leq 1$. Conditions $u(0, r, x) = u(l, r, x) = 0$; $u(z, r, -\pi\alpha) = u(z, r, \pi, \alpha) = 0$; $u(z, 1, x) = q(z, x)$, $x \in G$; $u_r(z, 1, x) = h(z, x)$, $x \in [-\pi\alpha, \pi\alpha] - G$; $u(z, \lambda, x) = g_1(z, x)$, $x \in G_1$; $u_r(z, \lambda, x) = h_1(z, x)$, $x \in [-\pi\alpha, \pi\alpha) - G_1$.
3. In spherical coordinates, where x is longitude, the domain is the ball $0 \leq r < 1$. Conditions: $u(1, x, \theta) = g(x, \theta)$ for $x \in G$; $u_r(1, x, \theta) = h(x, \theta)$, $x \in [-\pi, \pi] - G$.

The boundary conditions may also be nonhomogeneous for those variables with respect to which the integral transforms are carried out, but then the problem should first be reduced by known methods ([4], p. 101) to homogeneous boundary conditions.

Some nonperiodic problems indicated in ([1]) can be reduced to singular equations with the Cauchy kernel for an open contour; in this case, instead of the transformation (1), the Fourier integral is used.

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Note: Figure translations are in progress. See original paper for figures.