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Abstract

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ON THE THEORY OF MULTIPLICATION OF CAUSAL FUNCTIONS

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In the works of N. N. Bogolyubov and O. S. Parasyuk ([1−6]) a theory of multiplication of causal functions was constructed as applied to quantum field theory. However, the final formulas for the contributions obtained by these authors are inconvenient for investigating their analytic properties, since these formulas contain, in essence, unknown functions.

More precise expressions for the contributions can be obtained on the basis of certain combinatorial-topological notions. In particular, the consistent use of the notion of the incidence matrix ([2]) makes it possible, without loss of generality, to obtain formulas for the contributions that exactly reflect the structure of the diagrams.

Consider a strongly connected diagram G, consisting of V vertices and L internal lines. In the usual way we associate with it a product of causal functions, called the contribution of the diagram:

$$F(G) = \prod_{l=1}^{L} \Delta_l^c(x_\mu - x_\nu). \tag{1}$$

We shall first consider the case regularized according to Pauli–Villars, when all causal functions are replaced by the expressions

$$\operatorname{reg} \Delta^c(x_\mu - x_\nu) = \frac{1}{(2\pi)^4 i} \int e^{ikx} \int_0^\infty I(\alpha) e^{i\alpha(k^2 - m^2 + i\varepsilon)}\, d\alpha\, dk, \tag{2}$$

where $\varepsilon \to +0$,

$$I(\alpha) = 1 + \sum_{j=1}^{h} c_j e^{i\alpha(m^2 - M_j^2)}. \tag{3}$$

The coefficients c_j are chosen so that $I(\alpha)$ has a zero of order h at the point $\alpha = 0$.

The momentum representation of the regularized contribution is brought to the form

$$\operatorname{reg} F(G) = \pi^{2K}\left(\frac{1}{i}\right)^{K+L}\delta(\Sigma p)\int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_L \frac{\prod_{s=1}^{L} I(\alpha_s)}{M^2(\alpha)} \exp i\,\frac{D(\alpha)}{M(\alpha)}. \tag{4}$$

$$\varepsilon \to +0.$$

Here $M(\alpha)$ and $D(\alpha)$ are the characteristic determinants

$$M(\alpha) = \begin{vmatrix} A_{VV} & \cdots & A_{VL} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ A_{LV} & \cdots & A_{LL} \end{vmatrix}; \qquad D(\alpha) = \begin{vmatrix} A_{VV} & \cdots & A_{VL} & B_V \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ A_{LV} & \cdots & A_{LL} & B_L \\ B_U & \cdots & B_L & C \end{vmatrix}; \tag{5}$$

$$A_{rr'} = \sum_{k=1}^{V-1}\sum_{\sigma,\sigma'=1}^{V-1} \alpha_k\,[e_k^\sigma]\,[e_k^{\sigma'}]\,e_r^\sigma e_{r'}^{\sigma'} + \alpha_r\delta_{rr'} \quad (r, r' = V, \dots, L);$$

$$B_r = \sum_{k=1}^{V-1}\sum_{\sigma,\sigma'=1}^{V-1} \alpha_k\,[e_k^\sigma]\,[e_k^{\sigma'}]\,e_r^\sigma p_{\sigma'}; \tag{6}$$

$$C = \sum_{k=1}^{V-1}\sum_{\sigma,\sigma'=1}^{V-1} \alpha_k\,[e_k^\sigma]\,[e_k^{\sigma'}]\,p_\sigma p_{\sigma'} - \sum_{k=1}^{L}\alpha_k(m_k^2 - i\varepsilon).$$

The numbers e_k^σ are respectively equal to ± 1 or 0, depending on whether the line numbered k enters the vertex σ, leaves it, or bypasses it. These numbers form the above-mentioned incidence matrix of the diagram. $[e_k^\sigma]$ denote the algebraic cofactors of the elements e_k^σ of the diagram matrix obtained from the incidence matrix by deleting the last row and the last $K = L - V + 1$ columns.

It is readily obtained that $D(\alpha)/M(\alpha) = C - (B, A^{-1}B)$ is a quadratic form in the external momenta that is positive definite (in Sylvester' s sense) if and only if the parameters $\alpha_1, \dots, \alpha_{V-1} > 0$ and $\alpha_V \dots \alpha_L > 0$.

The contribution (4) obtained in this way is well defined only in the case of negative index of the diagram, equal to $\omega = 4K - 2L$. In this case passage to the β-representation gives the well-known result of Chisholm ([11])

$$\alpha_s = \lambda\beta_s; \qquad \sum_{s=1}^{L}\beta_s = 1;$$

$$F(G) = (i\pi^2)^K (L-2K-1)! \int_0^1 d\beta_1 \ldots \int_0^1 d\beta_L \, \delta\left(1 - \sum_{s=1}^{L} \beta_s\right) \frac{M(\beta)^{L-2K-2}}{D(\beta)^{L-2K}}. \tag{7}$$

If, however, the index of the diagram is nonnegative (a divergent diagram), passage to the β-representation gives

$$\operatorname{reg} F(G) = \frac{\pi^{2K} i^{K-2L}}{(2K-L)!} \sum_{i_1,\ldots,i_L=0}^{\omega/2+1} C_{i_1}^L \ldots C_{i_L}^L \int_0^1 d\beta_1 \ldots \int_0^1 d\beta_L \, \frac{\delta\left(1 - \sum_{s=1}^{L} \beta_s\right)}{M^2(\beta)} \times$$

$$\times \left[\sum_{s=1}^{L} \beta_s \left(m_{i_s}^2 - M_{i_s}^2\right) + \frac{D(\beta)}{M(\beta)}\right]^{\omega/2} \ln\left|\sum_{s=1}^{L} \left(m_{i_s}^2 - M_{i_s}^2\right) + \frac{D(\beta)}{M(\beta)}\right|. \tag{8}$$

It is known ([1,5]) that the Fourier transform of the contribution constructed in this way is defined only on functions which vanish, together with their derivatives, when any of their arguments coincide.

Extension of the functional to the entire space, taking account of the needs of quantum field theory, is achieved by applying the R-operation ([4,6,7]). As a result one obtains a sum of integrable expressions, the principal term of which we write here:

$$[1 - M(G)] \operatorname{reg} F(G) = -\frac{\pi^{2K}}{\omega!} \left(\frac{1}{-i}\right)^{K+L} \delta(\Sigma p) \int_0^1 d\tau \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L \, \frac{I(\alpha_1) \ldots I(\alpha_L)}{M^2(\alpha)} \times$$

$$\times \left(-i \sum_{\sigma,\sigma'=1}^{V-1} \Gamma_{\sigma\sigma'} p_\sigma p_{\sigma'}\right)^{\frac{\omega+1}{2}} \mathcal{H}_{\omega+1}\left[\tau \left(-i \sum_{\sigma,\sigma'=1}^{V-1} \Gamma_{\sigma\sigma'} p_\sigma p_{\sigma'}\right)^{1/2}\right] \exp i \, \frac{D_\tau(\alpha)}{M(\alpha)}. \tag{9}$$

$\mathcal{H}_n(x)$ denotes the ordinary Hermite polynomial,

$$\frac{D_\tau(\alpha)}{M(\alpha)} = \tau^2 \sum_{\sigma,\sigma'=1}^{V-1} \Gamma_{\sigma\sigma'} p_\sigma p_{\sigma'} - \sum_{k=1}^{L} \alpha_k \left(m_k^2 - i\varepsilon\right).$$

Let us also write out the expression obtained as a result of applying the operation $\Delta(G)$, contained in R:

$$\Delta(G)\,\mathrm{reg}\,F(G) = \left(\frac{1}{i}\right)^{L-m} \delta\left(\sum p\right) \int_0^\infty d\alpha_{m+1} \cdots \int_0^\infty d\alpha_L I(\alpha_{m+1}) \cdots I(\alpha_L)$$
$$\times \left\{ \prod_{j=1}^{m} \Lambda \left[\sum_{s=V}^{L} P_{js} a_s + \sum_{r=1}^{V-1} Q_{jr} p_r \right] \frac{1}{M^2(\alpha)} \exp i \frac{D(\alpha)}{M(\alpha)} \right\}_{\alpha_1 = \cdots = \alpha_m = 0}, \qquad (10)$$

$$a_s = -(A^{-1}B)_s;$$

$\Lambda(q)$ denotes the polynomial assigned to the line going inside the generalized vertex; P_{js}, Q_{jr} are numerical coefficients, equal to ± 1 or 0, simply connected with the matrix of the diagram.

Finally, we note that $\Delta(G)\,\mathrm{reg}\,F(G)$ is the contribution of the reduced diagram, obtained from the basic one by contracting the lines going inside the generalized vertex to a point, multiplied by certain polynomials in the momenta of the reduced lines.

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Note: Figure translations are in progress. See original paper for figures.