MATHEMATICS

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Abstract

Full Text

MATHEMATICS

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ON THE BEST UNIFORM APPROXIMATION OF DIFFERENTIABLE FUNCTIONS

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We introduce the following notation. H_ω is the class of continuous periodic functions $f(x)$, with period 2π, whose modulus of continuity

$$\omega(f;t) = \sup_{|x'-x''|\le t} |f(x') - f(x'')|$$

does not exceed the prescribed modulus of continuity $\omega(t)$. In particular, for $\omega(t) = Kt^\alpha$ $(0 < \alpha \le 1;\ 0 \le t \le \pi)$ the class H_ω is the class $KH^{(\alpha)}$ of periodic functions satisfying on the whole axis the Lipschitz condition of order α with constant K. $W^{(r)}H_\omega$ is the class of periodic functions $f(x)$, with period 2π, for which the derivative of order r, $f^{(r)}(x)$, belongs to the class H_ω. $W^{(r)}KH^{(\alpha)}$ is the class of functions $f(x)$ for which $f^{(r)}(x) \in KH^{(\alpha)}$. $E_n(f)$ is the best uniform approximation of the periodic function f by trigonometric polynomials of degree not exceeding n.

In my note (^{4}) it was shown that if the function $\omega(t)$ is convex upward, then

$$\sup_{f\in H_\omega} E_n(f) = \frac{1}{2}\omega\left(\frac{\pi}{n+1}\right) \quad (n = 0, 1, 2, \ldots). \tag{1}$$

Here, under the same assumption concerning $\omega(t)$, the exact value is given for the upper bound of the best approximations on the class $W^{(1)}H_\omega$. Namely, the following holds:

Theorem. If $\omega(t)$ is a convex upward modulus of continuity, then

$$\sup_{f\in W^{(1)}H_\omega} E_n(f) = \frac{1}{4}\int_0^{\frac{\pi}{n+1}} \omega(t)\,dt \quad (n = 0, 1, \ldots). \tag{2}$$

The proof of relation (2) is based on the following lemma:

Lemma. Let $f \in H_\omega$, where $\omega(t)$ is a convex upward modulus of continuity. Whatever the number $K > 0$, in the class $KH^{(1)}$ there is a function φ_0 such that, for any a and b,

$$\left|\int_a^b [f(x) - \varphi_0(x)]\,dx\right| \le \frac{1}{2}\max_{x\ge 0}\int_0^x |\omega(t) - Kt|\,dt. \tag{3}$$

The scheme of the proof of the lemma is as follows. Let $g(x)$ be a polygonal periodic function inscribed in f. The graph of $g(x)$ on the period $[0, 2\pi]$ consists of a finite number of rectilinear segments (links) $l_1, l_2, \dots, l_p$, whose slopes we shall denote respectively by $\gamma_1, \gamma_2, \dots, \gamma_p$.

Suppose that $\gamma_k > K$. Then there exist points $t'_k < x_k < t''_k$ such that the segment $[t'_k, t''_k]$ contains within itself the abscissas of the endpoints of the link l_k, and,

if we set

$$\tau_k(x) = g(x_k) + K(x - x_k), \qquad \Delta_k(x) = g(x) - \tau_k(x),$$
$$\int_a^b \Delta_k(x)\,dx = I_k(a, b), \tag{4}$$

then the following relations will be satisfied:

$$I_k(t'_k, x_k) = \min_{x\le x_k} I_k(x, x_k) < 0, \qquad I_k(x_k, t''_k) = \max_{x\ge x_k} I_k(x_k, x) > 0,$$

$$I_k(t'_k, t''_k) = 0.$$

Similarly, if $\gamma_k < -K$, then we find an interval $[t'_k, t''_k]$ containing the abscissas of the endpoints of l_k, and a point $x_k \in (t'_k, t''_k)$ such that, replacing K by $-K$ in the notation (4), we shall have

$$I_k(t'_k, x_k) = \max_{x\le x_k} I_k(x, x_k) > 0, \qquad I_k(x_k, t''_k) = \min_{x\ge x_k} I_k(x_k, x) < 0,$$

$$I_k(t'_k, t''_k) = 0.$$

From the line segments $\tau_k(x)$, constructed in the indicated way for the links with angular coefficients exceeding K in absolute value, and from the remaining links of the broken line $g(x)$, it is not difficult to construct a periodic polygonal function $\varphi(x) \in KH^{(1)}$ such that

$$\max_{a,b}\left|\int_a^b [g(x) - \varphi(x)]\,dx\right| \le 2\max_k |I_k(t'_k, x_k)|. \tag{5}$$

Now let us note that the following assertion is valid: whatever the function $f \in H_\omega$, where $\omega(t)$ is convex upward, if on the interval $[a,b]$ for the function

$$\Delta(x) = f(x) - f(c) - K(x-c) \qquad (a < c < b)$$

the relations

$$-\int_a^c \Delta(x)\,dx = \int_c^b \Delta(x)\,dx > 0,$$

are satisfied, then

$$M(f) = -\int_a^c \Delta(x)\,dx + \int_c^b \Delta(x)\,dx \le \frac{1}{2}\int_0^{b-a} [\omega(t) - Kt]\,dt. \tag{6}$$

Indeed, without loss of generality one may assume that $\Delta(x) < 0$ on (a,c) and $\Delta(x) > 0$ on (c,b). Then, defining the function $\rho(x)$ by the equality

$$\int_x^{\rho(x)} \Delta(t)\,dt = 0 \qquad (a \le x \le c,\ c \le \rho(x) \le b),$$

one can obtain the estimates

$$M(f) \le \frac{1}{2}\int_a^c |\Delta(\rho(t)) - \Delta(t)|(1-\rho'(t))\,dt \le$$

$$\le \frac{1}{2}\int_a^c \{\omega[\rho(t)-t] - K[\rho(t)-t]\}(1-\rho'(t))\,dt = \frac{1}{2}\int_0^{b-a} [\omega(t) - Kt]\,dt.$$

If the polygonal function $g(x)$ is inscribed in $f \in H_\omega$ and $\omega(t)$ is convex upward, then, as was already noted in the paper $(^4)$, $g \in H_\omega$.

Thus, from (5), using (6), we find that

$$\max_{a,b}\left|\int_a^b [g(x) - \varphi(x)]\,dx\right| \le \frac{1}{2}\max_{x\ge 0}\int_0^x [\omega(t) - Kt]\,dt.$$

It is known that in any continuous function $f(x)$ one can inscribe a polygonal function $g(x)$ so that $\max_x |f(x) - g(x)| < \varepsilon$, where ε may be prescribed arbitrarily small. Therefore, if f satisfies the conditions of the lemma, then it is easy to prove the existence of a function $\varphi_0 \in KH^{(1)}$, for which, for any a and b, inequality (3) is fulfilled.

Having completed the proof of the lemma, we note that, although in the present paper we shall not need this, a somewhat more general assertion holds, namely

$$\sup_{f\in H_\omega}\ \inf_{\varphi\in KH^{(1)}}\ \max_{a,b}\left|\int_a^b [f(x)-\varphi(x)]\,dx\right| = \frac{1}{2}\max_{x\geq 0}\int_0^x [\omega(t)-Kt]\,dt.$$

We now pass to the proof of the theorem. Let $f\in W^{(1)}H_\omega$. Then $f'\in H_\omega$, and, by the lemma, for any $K>0$ there exists a function $\varphi_0\in KH^{(1)}$ such that

$$\left|\int_{a_0}^{b_0}[f'(x)-\varphi_0(x)]\,dx\right| = \max_{a,b}\left|\int_a^b [f'(x)-\varphi_0(x)]\,dx\right| \leq \frac{1}{2}\delta(K),$$

where, for brevity, we have denoted

$$\delta(K)=\max_{x\geq 0}\int_0^x [\omega(t)-Kt]\,dt.$$

Set

$$\psi_0(x)=\int_c^x \varphi_0(t)\,dt,$$

where the value of c has been chosen from the condition

$$\int_{a_0}^{c}(f'(t)-\varphi_0(t))\,dt=\int_c^{b_0}(f'(t)-\varphi_0(t))\,dt.$$

Then

$$\max_x|f(x)-\psi_0(x)| = \left|\int_c^{t_0}[f'(t)-\varphi_0(t)]\,dt\right| \leq \frac{1}{4}\delta(K).$$

Since $\psi_0\in W^{(1)}KH^{(1)}$, by $(^{1,2})$

$$E_n(\psi_0)\leq K\frac{\pi^2}{8(n+1)^2},$$

and therefore

$$E_n(f)\leq \max_x|f(x)-\psi_0(x)|+E_n(\psi_0)\leq \frac{1}{4}\delta(K)+K\frac{\pi^2}{8(n+1)^2}.$$

If we take $K = K_n = \frac{n+1}{\pi}\omega\left(\frac{\pi}{n+1}\right)$, then

$$\delta(K_n) = \int_0^{\frac{\pi}{n+1}} [\omega(t) - K_n t]\, dt,$$

and we obtain

$$E_n(f) \leq \frac{1}{4}\int_0^{\frac{\pi}{n+1}} \omega(t)\, dt. \tag{7}$$

It remains to indicate, in the class $W^{(1)}H_\omega$, a function for which equality holds in (7). Such a function is, for example, the function $f_0(x)$, whose derivative f_0' has period $\frac{2\pi}{n+1}$, is odd, and is defined on

$$\left[-\frac{\pi}{2(n+1)}, \frac{\pi}{2(n+1)}\right]$$

by the equalities

$$f_0'(x) = \begin{cases} \frac{1}{2}\,\omega(2x), & \left(0 \leq x \leq \frac{\pi}{2(n+1)}\right), \\ -\frac{1}{2}\,\omega(-2x), & \left(-\frac{\pi}{2(n+1)} \leq x \leq 0\right). \end{cases}$$

The theorem is proved. Let us note the most important special case, when $\omega(t) = Kt^\alpha$ $(0 \leq t \leq \pi)$.

Corollary. For all $0 < \alpha \leq 1$,

$$\sup_{f\in W^{(1)}KH^{(\alpha)}} E_n(f) = \frac{K}{4(1+\alpha)}\left(\frac{\pi}{n+1}\right)^{1+\alpha} \qquad (n = 0, 1, 2, \ldots). \tag{8}$$

Let $W^1KH^{(\alpha)}_{[-1,1]}$ be the class of functions whose derivative satisfies, on the interval $[-1, 1]$, a Lipschitz condition of order α $(0 < \alpha \leq 1)$ with constant K, and let $E_n(f; -1, 1)$ be the best approximation of the function f by algebraic polynomials of degree n on this interval. Then, taking into account the limiting equality proved by S. N. Bernstein (^{3}), from (8) we immediately find that

$$\lim_{n\to\infty} (n+1)^{1+\alpha} \sup_{f\in W^{(1)}KH^{(\alpha)}_{[-1,1]}} E_n(f; -1, 1) = \frac{K\pi^{1+\alpha}}{4(1+\alpha)} \qquad (0 < \alpha \leq 1).$$

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REFERENCES

1. N. I. Akhiezer, M. G. Krein, DAN, **15**, 107 (1937).
2. J. Favard, Bull. de Sci. Math. **61**, 209 (1937).
3. S. N. Bernstein, DAN, **57**, 3 (1947).
4. N. P. Korneichuk, DAN, **140**, No. 4 (1961).

Note: Figure translations are in progress. See original paper for figures.