MATHEMATICS

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Abstract

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ON THE APPROXIMATION OF FUNCTIONS BY POLYNOMIALS SATISFYING NONLINEAR RELATIONS

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1°. Let a continuous real-valued function $f(q)$ be given on a compact set Q, and let a class K of real generalized polynomials of the form

$$y(q) = \sum_{k=1}^{n} a_k \varphi_k(q), \tag{1}$$

be specified, where $\{\varphi_k(q)\}_1^n$ is a system of n linearly independent functions continuous on Q. Suppose it is required to approximate uniformly the function $f(q)$ by polynomials (1) subject to given relations, which will be discussed below in 2°.

For each polynomial $y(q) \in K$, put

$$L_f(y) = \max_{q \in Q} |f(q) - y(q)|, \tag{2}$$

and by $E(y)$ denote the set of all points of maximum deviation of $y(q)$ from $f(q)$, i.e. the points $q \in Q$ at which $|f(q) - y(q)| = L_f(y)$.

It is known (^{1}) that the problem of best approximation (in the sense of P. L. Chebyshev) of the function $f(q)$ by polynomials from K is always solvable. Therefore we are entitled to set

$$\min_{y \in K} L_f(y) = \rho^*. \tag{3}$$

Passing to geometric language, we shall identify a polynomial $y(q) \in K$ with its representing point $(a_1, a_2, \dots, a_n)$ in the n-dimensional coefficient space R_n. Then the aggregate of all polynomials of best approximation for $f(q)$ may be interpreted as a bounded, closed, and convex set $V^* \equiv V_{\rho^*} \subset R_n$. By S_{ρ^*} we denote the boundary of V_{ρ^*}.

If $L > \rho^*$, then the equality $L_f(y) = L$ is, obviously, realized on a nonempty subset of polynomials from K, which form in R_n a closed surface S_L bounding the convex body V_L. It is clear that for $L' < L''$ we shall have

$$V_{L'} \subset V_{L''}, \qquad S_{L'} \cap S_{L''} = \Lambda. \tag{4}$$

2°. In order to impose a relation on the polynomials (1), let us single out in R_n a nonempty closed set Ω and define a narrower class of comparison polynomials $K_\Omega \subset K$, considering $y(q) \in K_\Omega$ if and only if $(a_1, a_2, \dots, a_n) \in \Omega$. In concrete cases the set Ω may be specified, for example, by inequalities

$$\alpha_j \leqslant \omega_j(y) \leqslant \beta_j \qquad (j = 1, 2, \dots, p), \tag{5}$$

where $\omega_j(y)$ are continuous functionals, and α_j, β_j are constant numbers ensuring the admissibility of the relations (5), i.e. the nonemptiness of Ω. In particular, this

may be the linear relations, which have also been studied repeatedly ($^{2-8}$),

$$\omega_j(y) \equiv \sum_{k=1}^{n} \delta_{jk} a_k = \beta_j \quad (j = 1, 2, \dots, p;\ p \leq n). \tag{6}$$

Definition. $y^*(q)$ is called an **extremal polynomial** of the function $f(q)$ in the class K_Ω, if $y^*(q) \in K_\Omega$ and for it

$$L_f(y^*) = \inf_{y \in K_\Omega} L_f(y) \equiv \rho_\Omega. \tag{7}$$

Let the set of the representing points of all extremal polynomials in the space R_n be denoted by V^*_Ω.

3°. From relations (4) and the closedness of the set Ω there follows the existence of a least value $L \geq \rho^*$, for which the intersection $\Omega \cap S_L$ is nonempty. In turn, hence there follows:

Theorem 1. *For every continuous function $f(q)$ on Q there exists at least one extremal polynomial $y^*(q)$, and*

$$\rho_\Omega = \min_{\Omega \cap S_L \neq \Lambda} L, \qquad V^*_\Omega = \Omega \cap S_{\rho_\Omega}. \tag{8}$$

Moreover, if Ω is convex, then V^*_Ω is also convex and lies in a supporting hyperplane for V_{ρ_Ω}. At the same time, on the compact set Q there exists a common basis (cf. (^{4b})) of joint deviation points of all extremal polynomials.

Theorem 2. *For uniqueness of the extremal polynomial* $y^*(q)$ *it is sufficient that the convex hull of the set* Ω *be strictly convex, contain no points of* V^* *in its interior, and that its surface be wholly contained in* Ω*.*

It is not difficult to extend from the linear case the criterion of B. A. Rymarenko (^{4a}).

Theorem 3. *In order that the polynomial* $y^*(q) \in K_\Omega$ *be extremal, it is sufficient —and, in the case of convexity of the set* Ω*, necessary—that for every polynomial* $y(q) \in K_\Omega$ *one have*

$$\min_{q \in E(y^*)} \{[y(q) - y^*(q)][f(q) - y^*(q)]\} \leq 0. \tag{9}$$

4°. Let Ω be convex and $y_0^* \in V_\Omega^*$. Then, if also $y^* \in V_\Omega^*$, it is necessary that

$$\min_{q \in E(y_0^*)} \{[y^*(q) - y_0^*(q)][f(q) - y_0^*(q)]\} = 0. \tag{10}$$

Conversely, if a polynomial $y^*(q) \in K_\Omega$ satisfies relation (10), then for sufficiently small $\lambda \geq 0$ the comparison polynomial $y_0^*(q) + \lambda[y^*(q) - y_0^*(q)]$ will be extremal.

The introduction of the parameter λ and the geometric considerations of items 1°–3° make it possible to apply the theory of the one-parameter problem, set forth in the author' s papers (^{9}). This helps to clarify questions concerning the general form of extremal polynomials.

Theorem 4. *Let* $y_0^*(q)$ *be one of the extremal polynomials. In order that* $y^*(q) \in K_\Omega$ *also be an extremal polynomial, it is sufficient—and, when* Ω *is convex, necessary—that the conditions*

$$N(y^* - y_0^*) \cap E(y_0^*) \neq \Lambda, \tag{11}$$

$$\inf_{q \in Q \setminus N(y^* - y_0^*)} \left\{ \frac{[f(q) - y_0^*(q)] \operatorname{sign}[y^*(q) - y_0^*(q)] + \rho_\Omega}{|y^*(q) - y_0^*(q)|} \right\} \geq 1, \tag{12}$$

be fulfilled, where $N(h)$ *denotes the set of all zeros of the polynomial* $h(q) \in K$ *on the compact set* Q*.*

Theorem 5. If the set Ω generates the linear relations (6), then the general form of all extremal polynomials is given by the formula

$$y^*(q) = y_0^*(q) + \lambda h(q), \tag{13}$$

where $y_0^*(q)$ is any fixed extremal polynomial; $h(q)$ is an arbitrary polynomial of the class K satisfying the conditions

$$\omega_j(h) = 0 \quad (j = 1, 2, \dots, p); \tag{14}$$

$$N(h) \cap E(y_0^*) \neq \Lambda; \tag{15}$$

$$\max_{q \in Q} |h(q)| = 1, \tag{16}$$

and λ is any value from the segment $[\lambda^-, \lambda^+]$, where

$$\lambda^- = \sup_{q \in Q \setminus N(h)} \left\{ \frac{[f(q) - y_0^*(q)] \operatorname{sign} h(q) - \rho_\Omega}{|h(q)|} \right\},$$

$$\lambda^+ = \inf_{q \in Q \setminus N(h)} \left\{ \frac{[f(q) - y_0^*(q)] \operatorname{sign} h(q) + \rho_\Omega}{|h(q)|} \right\}. \tag{17}$$

5°. The geometric approach set forth above opens the possibility of far-reaching applications. For example, it facilitates the analysis of approximation problems by polynomials with nonnegative coefficients, with fixed coefficients at the specified $\varphi_k(q)$, by regularly monotone (10) polynomials, etc., when Ω is known in advance to be convex. One may also, proceeding from the "other end" (cf. (8)), impose relations under which preassigned polynomials with equal $L_f(y)$ will turn out to be extremal.

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Note: Figure translations are in progress. See original paper for figures.

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