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**Abstract**

**Full Text**

## Reports of the Academy of Sciences of the USSR

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**MATHEMATICS**

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### ON THE DIVERGENCE OF FOURIER SERIES

*(Presented by Academician A. N. Kolmogorov on 4 XI 1960)*

1. N. N. Luzin <sup>(1)</sup> constructed a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx +$$

$$+ b_k \sin kx$$

with coefficients tending to zero, which diverges unboundedly almost everywhere on  $[0, 2\pi)$ .\* Subsequently, the analogous problem was considered for trigonometric series on which more and more restrictions were imposed. A. N. Kolmogorov <sup>(3)</sup> gave an example of a Fourier-Lebesgue series diverging almost everywhere, and somewhat later <sup>(4)</sup> an example of a Fourier series diverging at every point. In A. N. Kolmogorov's examples the conjugate series

$$\sum_{k=1}^{\infty} a_k \sin kx - b_k \cos kx$$

is not a Fourier series (see, for example, <sup>(5)</sup>). The next step was made by Hardy and Rogosinski <sup>(6)</sup>: they constructed a Fourier series diverging almost everywhere, whose conjugate is also a Fourier series.

In parallel, the study of sets of convergence and divergence of trigonometric series was carried out. As is known <sup>(7)</sup>, the sets of points of unbounded divergence of a sequence of continuous functions on a closed set are of type  $G_\delta$ . Herzog and Piranian <sup>(8)</sup> constructed, for a given set  $E$  of type  $G_\delta$  on the unit circle, a Taylor series which diverges unboundedly on  $E$  and converges on the complementary set; then S. B. Stechkin <sup>(9)</sup> constructed a trigonometric series which converges on a prescribed set  $E \subset [0, 2\pi)$  of type  $F_\sigma$  and diverges unboundedly on  $E_1 = [0, 2\pi) - E$ ; finally, Zeller <sup>(10)</sup> constructed a Fourier series with the properties indicated above.

In the present note a theorem is proved that is more precise than Zeller's theorem. From it, in particular, follows the existence of a pair of conjugate Fourier series diverging unboundedly at every point.

A formulation of a somewhat weaker theorem was published by me in <sup>(11)</sup>.

**2. Lemma 1.** Let  $[a, b] \subset [0, 2\pi)$ ,  $[c, d] \subset (a, b)$ , and  $\varepsilon > 0$ . Then there exists a trigonometric polynomial

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos kx + b_k \sin kx,$$

having the following properties:

$$1) \quad \|T\|_L = \int_0^{2\pi} |T(x)| dx < \varepsilon;$$

\* As S. B. Stechkin <sup>(2)</sup> showed, the trigonometric series constructed by N. N. Luzin in fact diverges at every point.

2) for all  $x \in [0, 2\pi) \setminus [a, b]$ ,

$$|S_k(x, T)| = \left| \frac{a_0}{2} + \sum_{j=1}^k a_j \cos jx + b_j \sin jx \right| < \varepsilon$$

$$|\tilde{S}_k(x, T)| = \left| \sum_{j=1}^k a_j \sin jx - b_j \cos jx \right| < \varepsilon, \quad k = 1, 2, \dots;$$

3) for every  $x \in [c, d]$  there is an index  $k_x$  such that  $|S_{k_x}(x, T)| > 1/\varepsilon$ .

This lemma is a slight modification of one result of A. N. Kolmogorov (see <sup>(12)</sup>, § 8.404) and is proved analogously.

For each natural  $n$  we construct the points  $x_l = \frac{2\pi l}{2n+1}$ ,  $l = 0, 1, \dots, 2n$ , and choose  $n_0$  so that

$$\frac{2\pi}{2n_0+1} < \min \left\{ \frac{c-a}{4}, \frac{d-b}{4} \right\} = \delta.$$

For each  $n \geq n_0$ , among the division points  $x_l$  we choose the point  $x_{l_0}$  nearest from the left to the point  $c$ , and the point  $x_{l_1}$  nearest from the right to the point  $d$ . From the Fejér polynomial  $K_{2n(n+1)}(x)$  of order  $2n(n+1)$  we form the polynomial

$$\varphi(x) = \frac{1}{m'} \sum_{l=l_0}^{l_1} K_{2n(n+1)}(x - x_l), \quad \text{where } m' = l_1 - l_0 + 1.$$

Choose  $\eta = \eta(n)$  so that for  $x_l - \eta \leq x \leq x_l + \eta$  the inequality  $K_{(2n+1)*}(x - x_l) \geq n(n+1)/2$  holds and the Dirichlet sum of order  $2n(n+1)$  is positive, i.e.  $D_{(2n+1)*}(x - x_l) > 0$ .

Put  $h = (a+c)/2$ ,  $e = (d+b)/2$ . Let  $x_{\nu_0}$  be the point nearest from the left to the point  $h$ , and  $x_{2\nu_1}$  the point nearest from the right to the point  $e$  among all points  $x_{2\nu}$ ,  $\nu = 0, 1, \dots, n$ . Consider the trigonometric polynomial

$$\psi(x) = \frac{1}{m''} \sum_{\nu=\nu_0}^{\nu_1} K_{m_\nu}(x - x_{2\nu}), \quad m'' = \nu_1 - \nu_0 + 1.$$

Choose the numbers  $m_{\nu_0}, \dots, m_{\nu_1}$  so that the following conditions are satisfied:  $m_{\nu_0} > (2n+1)^2$ ; to each  $x \in (x_j + \eta, x_{j+1} - \eta)$ ,  $2\nu_0 \leq j \leq 2\nu_1$ , there corresponds a natural  $k = k_x$  such that  $m_j \leq k < m_{j+1}$ ,  $\sin(k+1/2)(x_{2j} - x) \geq 1/3$ , and  $2k+1$  is divisible by  $2n+1$ . The possibility of such a choice of  $m_\nu$  was proved by A. N. Kolmogorov (see <sup>(12)</sup>).

Put

$$T(x) = \frac{\varphi(x) + \psi(x)}{(\ln n)^{1/2}}.$$

For sufficiently large  $n$  the polynomial  $T(x)$  will satisfy all the conditions of the lemma.

**Lemma 2.** Let  $M > 0$ . Then for every trigonometric polynomial

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos kx + b_k \sin kx$$

there exists a trigonometric polynomial of the form

$$t(x) = \sum_{k=Q}^R c_k \cos kx + d_k \sin kx,$$

where  $Q > M$ , such that for every  $x \in [0, 2\pi)$

$$|t(x)| \leq |T(x)|, \quad |\tilde{t}(x)| \leq |\tilde{T}(x)|, \quad (1)$$

$$\frac{1}{4} \sup_k |S_k(x, T)| \leq \sup_n |S_n(x, t)| < \sup_k \{ |S_k(x, t)| + |S_k(x, \tilde{T})| \}. \quad (2)$$

**Proof.** Put

$$t(x) = \frac{1}{2}(\cos Px + \sin 3Px)T(x),$$

where  $P = M + N + 1$ . Then

$$\tilde{t}(x) = \frac{1}{2}(\sin Px - \cos 3Px)T(x),$$

and property (1) is obvious.

Let us prove property (2). The following identities hold: for  $k \leq P$ ,

$$S_k(x, t) = \frac{1}{4} \{ \cos Px [T(x) - S_{P-k-1}(x, T)] + \sin Px [\tilde{T}(x) - S_{P-k-1}(x, \tilde{T})] \}; \quad (3)$$

for  $P \leq k \leq P + N$ ,

$$S_k(x, t) = \frac{1}{2} \{ \cos Px \cdot S_{k-P}(x, T) + S_{2P-k-1}(x, t) \}; \quad (4)$$

for  $P + N \leq k$ ,

$$S_k(x, t) = \frac{1}{2} \{ \cos Px \cdot T(x) + S_k(x, T \sin 3Px) \}. \quad (5)$$

The second term in equality (5) is expanded analogously to equalities (3) and (4).

Estimating (3), (4), and (5) in absolute value, we obtain the right-hand side of inequality (2).

Let us prove the left-hand side of inequality (2). Choose  $k_x$  so that

$$\sup_k |S_k(x, T)| = S_{k_x}(x, T).$$

Then

$$\cos Px \cdot S_{k_x}(x, T) = S_{P+k_x}(x, t) - S_{P-k_x-1}(x, t).$$

If  $|\cos Px| \geq \frac{1}{2}$ , then

$$|S_{P+k_x}(x, t) - S_{P-k_x-1}(x, t)| \geq \frac{1}{2} |S_{k_x}(x, T)| = |\sup_k |S_k(x, T)||.$$

Consequently, one of the terms is not less than  $\frac{1}{4} \sup_k |S_k(x, T)|$ . If, however,  $|\cos Px| \leq \frac{1}{2}$ , then  $|\sin 3Px| \geq \frac{1}{2}$ , and we may use the polynomial  $\sin 3PxT(x)$  for the estimate. The conjugate polynomial  $\tilde{t}(x)$  is considered analogously. The lemma is completely proved.

**3. Theorem.** *Let  $E \subset [0, 2\pi)$  be an arbitrary set of type  $F_\sigma$ . Then there exists a pair of conjugate Fourier series converging on  $E$  and diverging unboundedly on  $E_1 = [0, 2\pi) - E$ .*

**Proof.** For intervals satisfying the conditions

$$(c_k, d_k) \subset [a_k, b_k] \subset [0, 2\pi),$$

and numbers  $\varepsilon_k = 2^{-k}$ ,  $k = 1, 2, \dots$ , we can, relying on Lemmas 1 and 2, construct by induction a sequence of polynomials

$$T_k(x) = \sum_{n=p_k}^{q_k} a_n \cos nx + b_n \sin nx,$$

satisfying the conditions:

- 1)  $p_k > q_{k-1}$ ;
- 2)  $\|T_k\|_L < 2^{-k}$ ,  $\|\tilde{T}\|_L < 2^{-k}$ ;
- 3) for all  $m$  and  $x \in [0, 2\pi) - [a_k, b_k]$ ,

$$|S_m(x, T_k)| < 2^{-k}, \quad |\tilde{S}_m(x, T_k)| < 2^{-k};$$

- 4) for every  $x \in [c_k, d_k]$  there is a natural number  $p_x^{(k)} \geq p_x$  such that

$$|S_{p_x^{(k)}}(x, T_k)| > 2^k, \quad |\tilde{S}_{p_x^{(k)}}(x, T_k)| > 2^k.$$

Using the Herzog–Piranian–Zeller construction (see, for example, <sup>(5)</sup>), we can construct a Fourier series that satisfies the theorem.

The consequence of this theorem noted in item 1 could also have been obtained from A. N. Kolmogorov's polynomials (see, for example, <sup>(12)</sup>) and Lemma 2.

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*Note: Figure translations are in progress. See original paper for figures.*

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