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Abstract

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THEORY OF ELASTICITY

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ON THE CONSTRUCTION OF THE THEORY OF ELASTICITY

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General relations for nonlinearly elastic bodies were considered in ([1,2]). Below we consider questions of constructing a theory of elasticity of an isotropic body under small deformations, which, in certain simplest experiments (uniaxial tension–compression, pure shear, hydrostatic compression), obeys Hooke' s law. It is shown that, within the framework of the assumptions adopted, there exists the possibility of constructing a sufficiently broad class of relations of the theory of elasticity.

Let us consider the deformation of an isotropic elastic body. Suppose that, when the sign of the stresses is changed to the opposite one, the components of deformation likewise only change sign; in other words, under tension and compression the body behaves in a completely analogous manner. It is expedient to single out the class of such bodies from the totality of isotropic bodies and to call them normally isotropic.

For an elastic body there exists a deformation potential

$$\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}}, \tag{1}$$

where σ_{ij} are the stress components, ε_{ij} are the strain components.

In view of the assumptions made, in the general case the deformation potential may be represented in the form

$$U = U\left(|\sigma|, \Sigma_2, |\Sigma_3|\right), \tag{2}$$

where σ is the first invariant of the stress tensor; Σ_2, Σ_3 are, respectively, the second and third invariants of the stress deviator:

$$\sigma = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z),$$

$$\Sigma_2 = (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2), \tag{3}$$

$$\Sigma_3 = s_{x s_{y s}} z + 2\tau_{xy}\tau_{yz}\tau_{zx} - s_x\tau_{yz}^2 - s_y\tau_{zx}^2 - s_z\tau_{xy}^2,$$

where $s_x = \sigma_x - \sigma$.

Let us agree to understand by a simplest experiment the loading along a completely definite ray in the stress-strain space ([3]). Let us consider the restrictions imposed on the expressions for the relation $\sigma_{ij} - \varepsilon_{ij}$ of an isotropic elastic body by the principal simplest experiments. According to (1), from (2), (3) we obtain

$$\varepsilon_x = \frac{1}{3}\frac{\partial U}{\partial \sigma} + 2\frac{\partial U}{\partial \Sigma_2}(2\sigma_x - \sigma_y - \sigma_z) + \frac{\partial U}{\partial |\Sigma_3|}(\operatorname{sign}\Sigma_3)\left(s_{ys}z - \tau_{yz}^2 + \frac{1}{18}\Sigma_2\right), \ldots$$

$$\varepsilon_{xy} = 12\frac{\partial U}{\partial \Sigma_2}\tau_{xy} + 2\frac{\partial U}{\partial |\Sigma_3|}(\operatorname{sign}\Sigma_3)(\tau_{yz}\tau_{zx} - s_z\tau_{xy}), \ldots \tag{4}$$

The missing expressions are obtained by cyclic permutation of the indices. Suppose that volumetric compression is directly proportional to the action of the mean stress

$$\sigma = 3K\varepsilon, \qquad \varepsilon = \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z). \tag{5}$$

From (4) it follows that

$$3\varepsilon = \frac{\partial U}{\partial \sigma}. \tag{6}$$

From (5) and (6) we obtain

$$\frac{\partial U}{\partial \sigma} = \frac{\sigma}{K}, \qquad U = \frac{\sigma^2}{2K} + \Phi(\Sigma_2, |\Sigma_3|). \tag{7}$$

Suppose that under uniaxial tension—compression Hooke' s law holds,

$$\begin{gathered} \sigma_x = E\varepsilon_x, \qquad \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0, \\ \varepsilon_y \neq 0, \qquad \varepsilon_z \neq 0, \qquad \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0. \end{gathered} \tag{8}$$

From (4), (7), and (8) we obtain

$$\begin{aligned}
\varepsilon_x &= \frac{1}{9K}\sigma_x + 4\frac{\partial\Phi}{\partial\Sigma_2}\sigma_x + \frac{2}{9}\frac{\partial\Phi}{\partial|\Sigma_3|}(\operatorname{sign}\Sigma_3)\sigma_x^2,\\
\varepsilon_y &= \frac{1}{9K}\sigma_x - 2\frac{\partial\Phi}{\partial\Sigma_2}\sigma_x - \frac{1}{9}\frac{\partial\Phi}{\partial|\Sigma_3|}(\operatorname{sign}\Sigma_3)\sigma_x^2,\\
\varepsilon_z &= \frac{1}{9K}\sigma_x - 2\frac{\partial\Phi}{\partial\Sigma_2}\sigma_x - \frac{1}{9}\frac{\partial\Phi}{\partial|\Sigma_3|}(\operatorname{sign}\Sigma_3)\sigma_x^2.
\end{aligned} \tag{9}$$

It follows from (9) that under uniaxial tension—compression of an isotropic specimen one always has $\varepsilon_y = \varepsilon_z$.

Suppose that Poisson' s ratio is equal to ν; consequently, under uniaxial tension—compression one must have

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x. \tag{10}$$

The relation between Poisson' s ratio and the bulk modulus is easily found from expressions (5) and (8),

$$K = \frac{E}{3(1-2\nu)}.$$

The sum of the three equations (9) is identically equal to zero. Only one of them is independent; for example, the first, which we rewrite, taking into account that Hooke' s law (8) must hold:

$$\frac{\sigma_x}{E} = \frac{\sigma_x}{9K} + 4\frac{\partial\Phi}{\partial\Sigma_2}\sigma_x + \frac{2}{9}\frac{\partial\Phi}{\partial|\Sigma_3|}(\operatorname{sign}\Sigma_3)\sigma_x^2. \tag{11}$$

In the case of uniaxial tension—compression,

$$\Sigma_2 = 2\sigma_x^2, \qquad \Sigma_3 = \frac{2}{27}\sigma_x^3. \tag{12}$$

Equation (11) can be rewritten in terms of invariants in various ways. For example, multiplying expression (11) by σ_x and taking (12) into account, we find

$$2\frac{\partial\Phi}{\partial\Sigma_2}\Sigma_2 + 3\frac{\partial\Phi}{\partial|\Sigma_3|}|\Sigma_3| = \frac{1}{2}\frac{1}{E}\left(-\frac{1}{9K}\right)\Sigma_2. \tag{13}$$

The solution of equation (13) is written in the form

$$\Phi = \frac{1}{4}\left(\frac{1}{E} - \frac{1}{9K}\right)\Sigma_2 + \theta_1\left(\frac{\sqrt{\Sigma_2}}{\sqrt[3]{|\Sigma_3|}}\right). \tag{14}$$

Multiplying (11) by σ_x^2 and taking (12) into account, we find

$$54\,\frac{\partial\Phi}{\partial\Sigma_2}\,|\Sigma_3| + \frac{1}{18}\,\frac{\partial\Phi}{\partial|\Sigma_3|}\,\Sigma_2^2 = \frac{27}{2}\left(\frac{1}{E} - \frac{1}{9K}\right)\Sigma_3. \tag{15}$$

The solution of equation (15) is written in the form

$$\Phi = \frac{1}{4}\left(\frac{1}{E} - \frac{1}{9K}\right)\Sigma_2 + \theta_2\left(\Sigma_2^3 - 1458\,\Sigma_3^2\right). \tag{16}$$

Other expressions for the function Φ may be obtained analogously. Suppose that, under pure shear, Hooke' s law holds: $\tau_{xy} = G\varepsilon_{xy}$ (all other components of stress and strain are equal to zero). Analogously to (4), we find that in this case

$$\Phi = \frac{1}{12G}\,\Sigma_2 + \Psi(\Sigma_2, |\Sigma_3|), \tag{17}$$

where it must hold that

$$\frac{\partial\Psi}{\partial\Sigma_2} = \frac{\partial\Psi}{\partial|\Sigma_3|} = 0 \quad \text{when } \Sigma_3 = 0.$$

If we require that Hooke' s law hold simultaneously under uniaxial tension–compression, then, comparing expressions (14) and (17), we obtain

$$\frac{1}{E} - \frac{1}{9K} = \frac{1}{3G}, \qquad \frac{\partial\theta_1}{\partial\Sigma_2} = \frac{\partial\theta_1}{\partial|\Sigma_3|} = 0 \quad \text{when } \Sigma_3 = 0. \tag{18}$$

We shall satisfy relation (18) by setting, for example,

$$\theta_1 = \left(\frac{\Sigma_3}{\Sigma_2^{3/2}}\right)^{\alpha}, \qquad \alpha > 1.$$

Thus, the restrictions imposed by the simplest experiments do not determine the form of the relation $\sigma_{ij} - \varepsilon_{ij}$ in an isotropic elastic body. The relations of the generalized Hooke' s law used in the literature hold under the particular assumption $\Phi = \Phi(\Sigma_2)$, but it does not follow a priori from anywhere that the strain potential is independent of the third invariant of the stress deviator tensor.

Apparently, the following assertion is valid: the restrictions imposed by the results of any finite number of independent simplest experiments do not completely determine the strain potential and, consequently, the relations $\sigma_{ij} - \varepsilon_{ij}$.

If one assumes (5), that under simultaneous tension–compression in three mutually perpendicular directions the corresponding strains are superposed on one another (the condition of independence of the action of the principal stresses), then it is easy to prove that the generalized Hooke' s law holds. But this assumption is essentially equivalent to determining the results of the entire set of simplest experiments.

Our point of view is as follows: within the framework of quite definite properties of the material–elasticity, normal isotropy, the results of some set of simplest experiments, and so on–a quite definite form of the relation $\sigma_{ij} - \varepsilon_{ij}$ is realized.

The problem of finding the true relation σ_{ij}—ε_{ij} consists in determining the class of possible relations under the given constraints and in singling out the true relation σ_{ij}—ε_{ij} from the totality of all possible ones by solving the properly formulated variational problem.

The further construction of continuum mechanics rests on determining the relations between the stressed and deformed states for very different media, on constructing models of a continuum. At present, in essence, we possess a very narrow circle of satisfactory models, which have been constructed under the simplest assumptions.

In any branch of continuum mechanics, within the limits of the results of the simplest experiments that are practically available to us, one can construct a sufficiently large number of relations σ_{ij}—ε_{ij} for a complex stressed state. Therefore the development of criteria for determining the true relation between the stressed and deformed states of a medium from the class of possible ones appears to be a sufficiently important problem.

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Note: Figure translations are in progress. See original paper for figures.