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Abstract

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ON THE FORCES ACTING ON A SMALL PARTICLE IN AN ACOUSTIC FIELD IN AN IDEAL LIQUID

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A particle suspended in the field of a sound wave is acted upon by hydrodynamic forces from the liquid. In the linear approximation these forces are proportional to the velocity of the liquid ([1]) and on average do not lead to displacement of the particle. In problems of acoustic coagulation an essential role is played by the mean forces acting on a particle, arising as a result of second-order effects. The nature of these forces is related to the forces of radiation pressure in a sound wave ([2], [3]). The magnitude of the forces has been found only for particles located in an ideal liquid, in the work of King ([2]). The method of ([2]) consisted in the exact solution of the problem of the flow around a sphere in the field of a sound wave. Effects associated with viscosity and thermal conductivity, so far as we know, have not been taken into account.

Below we set forth a simple method that makes it possible to determine the magnitude of the mean forces acting on a particle in an arbitrary acoustic field, when the dimensions of the particle are much smaller than the wavelength of sound. In this article we shall consider the case of an ideal liquid.

The magnitude of the force is equal to the mean flux of momentum through any closed surface in the liquid surrounding the sphere:

$$F_i = -\oint \Pi_{ik}\, df_k.$$

Here Π_{ik} is the tensor of momentum-flux density in an ideal liquid: $\Pi_{ik} = p\delta_{ik} + \rho v_i v_k$. Let us take as this surface a sphere with radius much larger than λ, the wavelength of sound. The velocity potential φ in the linear approximation has the form of a sum of the incident wave and the wave scattered by the sphere:

$$\varphi(\mathbf{r}, t) = \varphi_p(\mathbf{r}, t) + \varphi_P(\mathbf{r}, t). \qquad (1)$$

At the same time, from Euler' s equations it follows ($p = p_0 + p'$)

$$p' = -\rho\frac{\partial\varphi}{\partial t} - \rho\frac{v^2}{2} + \frac{\rho}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2. \tag{2}$$

Since $\overline{\partial\varphi/\partial t} = 0$, for the mean force we obtain

$$F_i = -\oint\left\{\left[-\rho\frac{\overline{v^2}}{2} + \frac{\rho}{2c^2}\overline{\left(\frac{\partial\varphi}{\partial t}\right)^2}\right]\delta_{ik} + \rho\,\overline{v_i v_k}\right\}df_k. \tag{3}$$

Thus, in order to calculate the mean force with accuracy up to terms of second order in the velocity, it is sufficient to find the solution of the linear scattering problem.

In the wave zone the potential of the scattered wave can be expanded in multipoles:

$$\varphi_P = \frac{a(t-r/c)}{r} - \frac{(\dot{\mathbf{A}}(t-r/c),\mathbf{n})}{rc} + ... \tag{4}$$

In turn, the coefficients in (4) for particles with dimensions $R \ll \lambda$ are determined from the solution of the problem of flow past the body by an incompressible fluid (see ([1])). Indeed, for $r \ll \lambda$ the wave equation passes into the equation of an incompressible fluid $\Delta\varphi_p = 0$, whose decreasing solutions are

$$\varphi_p = \frac{a(t)}{r} - \frac{(\mathbf{A}(t),\mathbf{n})}{r^2} + ... \tag{4'}$$

For potential flow of an ideal fluid past a stationary sphere, the second term has the form

$$-(\mathbf{v}_p(t),\mathbf{n})\frac{R^3}{2r^2}, \tag{5}$$

where $\mathbf{v}_p(t)$ is the velocity in the incident wave at the location of the sphere. The first term corresponds to an "ejection" of mass due to compression of the gas in the incident wave and is equal to

$$a(t) = -\frac{R^3}{3\rho}\dot{\rho}_p(t). \tag{6}$$

Expressions (5) and (6) refer only to an immobile solid sphere. In the general case the sphere is partially carried along by the fluid, acquiring a velocity $\mathbf{u}(t)$:

$$\mathbf{u}(t) = \frac{3\rho}{\rho+2\rho_0}\mathbf{v}_p(t),$$

where ρ_0 is the density of the sphere. Therefore, if ρ_0 is comparable with the density of the fluid, in (5) one should substitute the relative velocity $\mathbf{v}_p - \mathbf{u}$. Taking into account the compressibility of the sphere itself gives, instead of (6),

$$a(t) = -\frac{R^3}{3\rho}\dot{\rho}_p(t)\left(1 - \frac{c^2\rho}{c_0^2\rho_0}\right). \tag{6'}$$

Collecting the results, we find for the potential of the wave scattered by the sphere (4) the expression

$$\varphi_p = -\frac{R^3}{3\rho r}\dot{\rho}_p f_1 - \frac{R^3}{2} f_2 \operatorname{div}\left(\mathbf{v}_p \frac{1}{r}\right), \tag{7}$$

where $f_1 = 1 - c^2\rho/c_0^2\rho_0$, $f_2 = 2(\rho_0 - \rho)/(2\rho_0 + \rho)$.

As was shown in ([2]), the magnitude of the mean force in a standing wave proves to be greater than in a plane traveling wave. In the first case, in the quadratic expression for the force (3) the interference terms between the incident and the scattered waves are essential, whereas for a traveling wave the magnitude of the momentum transmitted to the particle by the wave is determined only by the momentum carried away by the scattered sound. Indeed, using the relations between quantities in a plane traveling wave, we rewrite (3) in the following form:

$$F_x = -\rho \oint \left[\overline{v_p v_p}(1 + \cos\theta) + \overline{v_p^2}\cos\theta\right] df \tag{8}$$

(the x-axis is chosen in the direction of propagation of the incident wave, v_p is the projection of the velocity of the scattered wave in the direction of the radius of the sphere). On the other hand, in the absence of dissipation the mean energy flux through a closed surface is equal to zero. Expanding the expression for the energy flux $Q = \rho v\,(v^2/2 + w)$ ([1]) up to terms of second order, we find

$$\oint Q\,df = \rho c \oint \left[\overline{v_p v_p}(1 + \cos\theta) + \overline{v_p^2}\right] df = 0. \tag{9}$$

Whence

$$F_x = \rho \oint \overline{v_p^2}(1 - \cos\theta)\,df. \tag{8'}$$

For a monochromatic wave we obtain

$$F = \frac{4\pi I}{9c} R^2 (kR)^4 \left[f_1^2 + f_1 f_2 + \frac{3}{4} f_2^2\right], \tag{10}$$

where $I = \rho c u_0^2/2$ is the mean energy-flux density in the incident wave.

Formula (8′) is valid only in a plane traveling wave. In an arbitrary acoustic field, the principal contribution to the expression for the force (3) is given by the interference terms between the incident and scattered radiation, which leads to forces much larger than (10). Restricting ourselves in (3) to these terms, we find

$$F_i = -\oint \left\{ \left[-\rho\,\overline{(v_p v_n)} + \frac{c^2}{\rho}\,\overline{\rho_n \rho_p} \right] \delta_{ik} + \rho\left(\overline{v_{ip} v_{kn}} + \overline{v_{in} v_{kp}}\right) \right\} df_k.$$

Passing to integration over the volume, and using Euler’ s equations and the potential character of the flow, we obtain:

$$\mathbf{F} = -\int_V \overline{\mathbf{v}_n \left(\rho\,\mathrm{div}\,\mathbf{v}_p + \frac{\partial \rho_p}{dt}\right)}\,dV = -\rho \int_V \overline{\mathbf{v}_n \left(\Delta\varphi_p - \frac{1}{c^2}\frac{\partial^2 \varphi_p}{\partial t^2}\right)}\,dV; \quad (11)$$

but, according to (7),

$$\Delta\varphi_p - \frac{1}{c^2}\frac{\partial^2 \varphi_p}{\partial t^2} = \frac{4\pi}{3\rho} f_1 R^3 \dot{\rho}_n\,\delta(\mathbf{r}) + 2\pi f_2 R^3\,\mathrm{div}(\mathbf{v}_n \delta(\mathbf{r})).$$

Substituting this into (11) and introducing $U(\mathbf{r})$, the potential of the forces $\mathbf{F} = -\nabla U$, we obtain

$$U = 2\pi R^3 \rho \left\{ \frac{\overline{p_n^2}}{3\rho^2 c^2} f_1 - \frac{\overline{v_n^2}}{2} f_2 \right\} \quad (12)$$

($\overline{p_n^2}$ and $\overline{v_n^2}$ are the mean values of the square of the pressure and velocity oscillations in the wave at the location of the particle). The force in a standing wave $\varphi_n = -(u_0/k)\cos\omega t \cos kx$ is

$$F = 4\pi \overline{E} R^2 (kR) \sin 2kx \left\{ \frac{\rho_0 + \frac{2}{3}(\rho_0 - \rho)}{2\rho_0 + \rho} - \frac{1}{3}\frac{c^2 \rho}{c_0^2 \rho_0} \right\} \quad (13)$$

($\overline{E}$ is the mean density of sound energy). Formulas (10), (13) coincide with the results of [2,3].

In the field $U(\mathbf{r})$, particles gather near the minimum of the potential energy (12). The equilibrium distribution of the density of suspended particles $n(\mathbf{r})$ follows Boltzmann’ s formula

$$n(\mathbf{r}) \sim \exp\{-U(\mathbf{r})/kT\}. \tag{14}$$

At places of maximum density, further coagulation of the particles is facilitated. At ordinary sound intensities $I \sim 0.1$ W/cm^2 and $\omega \sim 10^4$ s^{-1}, the gathering of particles at the nodes or antinodes of the wave plays an essential role. The width of the region Δ in which the particles concentrate, for example, in a standing wave (13), is of the order*

$$(k\Delta)^2 = \frac{3kT}{10\pi R^3 \overline{E}}.$$

(For $\overline{E} \simeq 4 \cdot 10^{10}$ erg/cm^3, $\nu \simeq 1$ kHz, $\Delta \simeq 1.5 \cdot 10^{-6} R^{-3/2}$.) The time in which the distribution (14) is established is equal to $\tau \sim \eta (kR)^{-2} (\overline{E})^{-1}$, where η is the viscosity of the medium.

Let us emphasize that (12) applies to any field, with the exception of one close to a plane traveling wave. In particular, in a spherical converging or diverging—

* At very small distances between particles, or near a wall, the forces of interaction between particles and wit

converging wave

$$U(r) = \frac{\overline{Q} R^3}{2c} \left\{ \frac{f_1}{3r^2} - \frac{f_2}{2} \left(\frac{1}{r^2} + \frac{1}{k^2 r^4} \right) \right\} \tag{15}$$

($\overline{Q}$ is the power of the source).

When $f_2 > 0$, $f_1 > {}^3/_2 f_2$, the particles either gather at the center or go off to infinity, depending on the distance from the center; for $f_2 > 0$, $f_1 < {}^3/_2 f_2$ there is a fall toward the center; if $f_1 < {}^3/_2 f_2$, $f_2 < 0$, (15) has a minimum at distances of the order of the wavelength; finally, if $f_1 > {}^3/_2 f_2 > 0$, the particles are driven out to infinity. Of course, at very large distances from the center (15) becomes small, the spherical wave turns into a plane wave, and the principal role is played by the forces (10):

$$U(r) = \pm \frac{\overline{Q} R^2}{9cr} (kR)^4 \left\{ f_1^2 + f_1 f_2 + \frac{3}{4} f_2^2 \right\} \tag{16}$$

(the plus sign is for a diverging wave, the minus sign for a converging wave).

The values (16) and (15) become comparable when $r \sim R(kR)^{-4}$. For the applicability of the results presented, it is necessary that the condition $\sqrt{\eta/\rho\omega} \ll R$ be satisfied.

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Note: Figure translations are in progress. See original paper for figures.