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Abstract

Full Text

MATHEMATICS

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ON THE DIRICHLET PROBLEM AND THE FIRST MIXED PROBLEM FOR A PARABOLIC EQUATION

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I. Let us consider the equation

$$\frac{\partial u}{\partial t} - L_0(x,t,D)u - L_1(x,t,D)u = f(x,t), \tag{1}$$

where $x = (x_1, \dots, x_n)$; D is the differentiation operator with respect to x; $L_0(x,t,D)$ and $L_1(x,t,D)$ are linear differential operators: $L_0(x,t,D)$ is a homogeneous operator of order $2p$, while $L_1(x,t,D)$ is an operator of order $(2p-1)$.

We shall assume that

$$\inf_{\substack{(xt)\in Q \\ |\alpha|=1}} |\operatorname{Re} L_0(x,t,i\alpha)| = \delta_0 > 0, \tag{2}$$

$$|\alpha|^2 = \sum \alpha_i^2;$$

Q is the domain of variation of (x,t); δ_0 is some number.

In the domain Q, bounded by a sufficiently smooth closed surface Γ, it is required to find a solution of equation (1) satisfying on Γ the following conditions (the Dirichlet problem):

$$u\Big|_\Gamma = \frac{\partial u}{\partial n}\Big|_\Gamma = \dots = \frac{\partial^{p-1} u}{\partial n^{p-1}}\Big|_\Gamma = 0, \tag{3}$$

where n is the interior normal lying in the plane $t = \text{const}$ to the surface of intersection of this plane with the surface Γ.

We note that the class of equations (1) under consideration includes not only parabolic equations, but also "backward-parabolic" equations, since in formula (2) the sign of $\operatorname{Re} L_0(x,t,i\alpha)$ is not fixed. For simplicity in formulating the results of item I, we shall assume that the planes $t = \text{const}$ touch the surface Γ only at two points $B = (x^B, t^B)$ and $H = (x^H, t^H)$ ("upper" and "lower"); at the remaining points of Γ (points of the "lateral surface") the tangent plane makes with the axis Ot an angle $\gamma \neq \pi/2$. Let

$$t = t^H + \varphi_H(x_1 - x_1^H, \dots, x_n - x_n^H), \qquad t = t^B + \varphi_B(x_1 - x_1^B, \dots, x_n - x_n^B) \quad (4)$$

be the local equations of Γ in neighborhoods of the lower and upper points, respectively. Obviously, $\varphi_H(x - x^H) \geq 0$, while $\varphi_B(x - x^B) \leq 0$ for sufficiently small $|x - x^H|$ and $|x - x^B|$.

Theorem 1. For any function $f(x,t) \in \mathcal{L}_2(Q)$, the Dirichlet problem (1), (3) is uniquely solvable in the space

$$\mathcal{H} = W^{(1,\,2p,\dots,2p)}_{t,x_1,\dots,x_n,2}(Q) \cap \overset{0}{W}{}^{(0,\,p,\dots,p)}_{t,x_1,\dots,x_n,2}(Q),$$

if $\varphi_H(x) = O(|x|^{2p})$ and $\varphi_B(x) = O(|x|^{2p})$ as $|x| \to 0$.

For the solution of this problem the estimate

$$\|u\|_H \leq C\|f\|_{L_2(Q)},$$

is valid, where the constant C depends only on the boundary of the domain Q and on the coefficients of equation (1).

Theorem 2. If in condition (2) $\operatorname{Re} L_0(x,t,i\alpha) < 0$ (parabolicity), then in Theorem 1 the restriction on the order of $\varphi_n(x)$ may be dropped; if $\operatorname{Re} L_0(x,t,i\alpha) > 0$ (inverse parabolicity), then the restriction on the order of $\varphi_v(x)$ may be dropped.

The space $W^{1,2p,\dots,p}_{t,x_1,\dots,x_n,2}(Q)$ is the space of Slobodetskii ([1]), and the space

$$\overset{0}{W}{}^{(0,p,\dots,p)}_{t,x_1,\dots,x_n,2}(Q)$$

is the Hilbert space obtained by completing the set $C_0^\infty(Q)$ (the set of infinitely differentiable functions in Q with compact support) with respect to the norm

$$\|u\|^2_{W^{(0,p,\dots,p)}_{t,x_1,\dots,x_n,2}} = \int_Q \cdots \int \left(u^2 + \sum_{s_1+\dots+s_n=p} \left| D^{s_1}_{x_1} \cdots D^{s_n}_{x_n} u \right|^2 \right) dx\, dt.$$

Let us note that for the heat equation ($p = 1$) the conditions on the orders of $\varphi_n(x)$ and $\varphi_v(x)$ can be weakened by virtue of Petrovskii' s theorem ([2]); namely, one may require that for sufficiently small $|x|$, $\varphi_n(x) \geq \varphi_0(x)$, $\varphi_v(x) \leq -\varphi_0(x)$, where $\varphi_0(x)$ is the solution of the equation $|x|^2 = 4\varphi_0 \ln|\ln \varphi_0|$.

The proof of Theorems 1 and 2 is based on the following auxiliary propositions:

Lemma 1. Let

$$\Omega = (|x - x^n| \leq 1, \quad t^n + \varphi_n(x - x^n) \leq t \leq t^n + \varphi_n(x - x^n) + 1), \tag{5}$$

$v(x,t) \in L_2(\Omega)$, $v_t(x,t) \in L_2(\Omega)$, $v(x,t)|_{t=t^n+\varphi_n(x-x^n)} = 0$. Then for any function $\varphi_n(x - x^n) \geq 0$ for $|x - x^n| \leq 1$ we have

$$\iint_\Omega \frac{v^2(x,t)}{t^2}\,dx\,dt \leq 4 \iint_\Omega (v_t(x,t))^2\,dx\,dt.$$

Lemma 2. For every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that all functions $v(x,t)$ described in Lemma 1 and having $v_{x_i} \in L_2(\Omega)$, $i = 1, \dots, n$, satisfy the inequality

$$\iint_\Omega \frac{v^2(x,t)}{t^2}\,dx\,dt \leq$$

$$\leq \varepsilon \iint_\Omega (v_t(x,t))^2\,dx\,dt + C(\varepsilon)\left(\iint_\Omega (v(x,t))^2\,dx\,dt + \iint_\Omega |\nabla u|^2\,dx\,dt\right),$$

provided only that $\varphi_n(x - x^n)$ in the domain $|x - x^n| \leq 1$ vanishes only at a finite number of points ($C(\varepsilon)$ does not depend on the choice of the function $v(x,t)$).

Lemma 3. Let a function $u(x,t) \in \mathcal{H}(\Omega)$ be given in the domain Ω (5). Then it can be continued into the domain

$$\Omega_1 = (|x - x^n| \leq 1, \quad 0 \leq t \leq t^n + \varphi_n(x - x^n)) \tag{6}$$

so that the newly obtained function in $\Omega_1 + \Omega$ belongs to $\mathcal{H}(\Omega_1 + \Omega)$. Moreover, the following estimates hold:

$$\iint_{\Omega_1} |D^{2p-s}u|^2\,dx\,dt \leq \frac{C_s}{m^{2s-1}} \sum_{r=1}^{2p-s} \iint_\Omega \frac{|D^r u|^2}{\psi_n^{2(2p-s)-2r}} \left(\frac{|x|}{\psi_n}\right)^{4p-2s-1} dx\,dt, \tag{7}$$

$s = 0, 1, \dots, 2p - 1$,

$$\iint_{\Omega_1} |u|^2 dx\,dt \leq \frac{C_{2p}}{m^{4p-1}} \iint_{\Omega^\varepsilon} |u|^2 \left(\frac{\psi_{\text{H}}}{|x|}\right) dx\,dt, \tag{8}$$

where $C_k,\ k=0,\dots,2p$ are certain constants; m is an arbitrary positive constant; ψ_{H} is the solution with respect to ρ of the equation $t-t^{\mathrm{H}}=\varphi_{\mathrm{H}}(\rho\omega_1,\dots,\rho\omega_n)$, with $\omega_1^2+\dots+\omega_n^2=1$.

Lemma 1 is a generalization of Theorem 254 of the book ([3]) to the multidimensional case and can be proved by a similar method. In the proof of Lemma 2 some results of Khinchin and Marcinkiewicz on the theory of trigonometric series are used.

Lemma 3 is proved with the aid of a special adaptation of the Whitney-Hestenes method ([4]) for extending functions across a curvilinear boundary of a domain in the case when straightening the boundary is undesirable (here this is connected with the nonequivalence of the variables t and $x_1,\dots,x_n$).

Lemmas 1, 2, 3 are also valid for a neighborhood of the point $B=(x^{\mathrm{B}},t^{\mathrm{B}})$; in this case the domains Ω and Ω_1 must be situated below the plane $t=t^{\mathrm{B}}$.

The proof of Theorem 1 is based on Theorem 3 on an a priori estimate for a smooth solution of equation (1) under condition (3).

Theorem 3. If problem (1), (3) has a sufficiently smooth solution $u(x,t)$ and the conditions of Theorem 1 on the orders $\varphi_{\mathrm{H}}(x)$ and $\varphi_{\mathrm{B}}(x)$ as $|x|\to 0$ are satisfied, then

$$\|u\|_{\mathcal{H}(Q)}\le C\|f\|_{\mathcal{L}2(Q)}+C_1\|u\|_{\mathcal{L}2(Q)}. \tag{9}$$

Let us illustrate the proof of Theorem 3 on the example of the one-dimensional heat equation

$$\mathcal{L}u\equiv u_t-(a(x,t)u_x)_x=f(x,t). \tag{10}$$

We multiply both sides of (10) by u_t and integrate over the domain Q. After integration by parts we obtain

$$\|u_t\|^2_{\mathcal{L}2(Q)}-\frac{1}{2}\iint_Q a_t u_x^2\,dx\,dt+\frac{1}{2}\int_{\Gamma_1-\Gamma_2} a u_x^2\,dx=(f,u_t)_{\mathcal{L}2(Q)}, \tag{11}$$

where Γ_1 and Γ_2 are respectively the upper and lower parts of the contour Γ (the t-axis is directed upward). From (11), for any $\varepsilon_1>0$ it follows that

$$\|u_t\|^2_{\mathcal{L}2(Q)}\le\varepsilon_1(\|u_x\|^2_{\mathcal{L}4(Q)}+\|u_t\|^2_{\mathcal{L}2(Q)})+\frac{A}{2}\int_{\Gamma_1+\Gamma_2} u_x^2\,dx+C_1(\varepsilon_1)\|f\|^2+C_2(\varepsilon_1)\|u\|^2. \tag{12}$$

Splitting Γ_2 by the point $H=(0,0)$ into two parts Γ_{21} and Γ_{22} with equations $x=\psi_1(t)$ and $x=\psi_2(t)$, respectively ($\psi_i(t)=O(\sqrt{t}),\ \psi_i'(t)=O(1/\sqrt{t})$ as

$t \to 0;\ 0 \le \psi_1(t) \le b_1,\ -b_2 \le \psi_2(t) \le 0,\ \psi_1(a_1) = b_1,\ \psi_2(a_2) = -b_2$), we obtain

$$\int_{\Gamma_2} u_x^2\,dx \le C\left(\int_0^{a_1} \frac{u_x^2}{\sqrt{t}}\bigg|_{\Gamma_{21}} dt + \int_0^{a_2} \frac{u_x^2}{\sqrt{t}}\bigg|_{\Gamma_{22}} dt\right) = C\iint_{\Omega_{11}+\Omega_{12}} \frac{\partial}{\partial x}(u_x^2)\frac{dx\,dt}{\sqrt{t}}, \tag{13}$$

where Ω_{11} and Ω_{12} are the right and left parts of the domain Ω_1 of Lemma 3. (We note that the extension of the function $u(x,t)$ (Lemma 3) to the domain $\Omega_{11} + \Omega_{12}$ may be assumed to be identically zero, together with a sufficient number of derivatives, for $x = b_1$ and $x = -b_2$.) From (13), with the aid of (7), (8), and Lemma 2, for any $\varepsilon_2 > 0$ we obtain

$$\int_{\Gamma_2} u_x^2\,dx \le 2C\iint_{\Omega} \frac{u_{xx}u_x}{\sqrt{t}}\,dx\,dt \le \varepsilon_2\|u_{xx}\|^2_{\mathcal{L}2(Q)} + C_3(\varepsilon_2)\iint_{\Omega}\frac{u^2}{t^2}\,dx\,dt \le$$

$$\le \varepsilon_2\|u_{xx}\|^2_{\mathcal{L}2(Q)} + \varepsilon C_3(\varepsilon_2)\|u_t\|^2_{\mathcal{L}2(Q)} + C(\varepsilon)C_3(\varepsilon_2)(\|u\|^2_{\mathcal{L}2(Q)} + \|u_x\|^2).$$

Estimating similarly the integral over Γ_1, from (12) and equation (10), after choosing sufficiently small $\varepsilon_1, \varepsilon_2$, and then ε, we obtain the required estimate (9). In the multidimensional case the arguments remain essentially similar to those given above, but in the appropriate place one must use the a priori estimates for elliptic operators from work [5] for an equation of the 2nd order and from [6] for an equation of arbitrary order.

The proof of existence and uniqueness of the solution of the Dirichlet problem (1), (3) is carried out according to the following plan: first one constructs a generalized solution from
${}^0W^{(0,p,\ldots,p)}_{(t,x_1,\ldots,x_n,2)}(Q)$ (the construction of this solution is carried out by a method analogous to Vishik' s method [7] for solving a mixed problem), then its smoothness is proved under the corresponding smoothness of the right-hand side $f(x,t)$. In doing this, to prove smoothness inside Q one should use the results of [8] or [9], in a neighborhood of nonhorizontal parts of the boundary Γ—the results of [9], and in neighborhoods of the points H and B—Lemmas 1 and 2.

After this, uniqueness of the smooth solution is proved and, thanks to Theorem 3 on the a priori estimate, by means of passage to the limit in the integral identity defining the generalized solution, Theorem 1 is finally proved.

II. Let equation (1) be parabolic ($\operatorname{sign}\operatorname{Re} L_0(x,t,i\alpha) = 1$ in (2)), and let the boundary Γ of the domain Q consist of three parts: in the upper and lower parts, of pieces T_{v} and T_{n} of planes (the points B and H of the preceding item are replaced by the planes $T_{\text{v}} : t = t^B$ and $T_{\text{n}} : t = t^H$) and of the lateral surface Γ_{b} ($\Gamma = T_{\text{v}} + T_{\text{n}} + \Gamma_{\text{b}}$); the part T_{n} may be absent

(Theorem 2). Γ_b is assumed to be such that in some neighborhood U_0 of any of its points (x_0, t_0) the equation of Γ_b can be represented in a form uniquely solved with respect to one of $x = (x_1, \dots, x_n)$. It is convenient to introduce in U_0 local coordinates (ξ, τ), $\xi = (\xi_1, \dots, \xi_n)$, so that the origin of coordinates lies at the point (x_0, t_0), ξ_1 changes in the direction of n, τ in the direction of t, and the remaining $\xi_2, \dots, \xi_n$ in the plane tangent to Γ_b at the point (x_0, t_0).

We shall say that Γ_b at (x_0, t_0) satisfies the Lipschitz condition of order α, $0 < \alpha \leq 1$, if the local equation of Γ_b in U_0 has the form
$\tau = C_1|\xi_1|^{1/\alpha} + o(\xi_1^{1/\alpha})$, $C_1 \neq 0$.

Consider the first mixed problem for equation (1), i.e., the problem of finding such a solution of (1) in Q that

$$u|_{\Gamma_b} = \dots = \left.\frac{\partial^{p-1} u}{\partial n^{p-1}}\right|_{\Gamma_b} = 0, \qquad u|_{T_n} = 0 \tag{14}$$

(the condition on T_n is absent if T_n degenerates into the point H).

Theorem 4. *Problem (1)–(14) is uniquely solvable for any $f(x,t) \in \mathcal{L}_2(Q)$ in the space $\mathcal{H}(Q)$ (Theorem 1), if Γ_b satisfies the Lipschitz condition of order γ, $\gamma \geq 1/2p$, at each of its points.*

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Note: Figure translations are in progress. See original paper for figures.