



Soviet-era science, translated into English

MATHEMATICS

1961

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.37104>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

G. I. KATS

A GENERALIZATION OF THE GROUP DUALITY PRINCIPLE

(Presented by Academician A. N. Kolmogorov on 2 XII 1960)

1. Let \mathfrak{G} be a locally compact commutative group. According to Pontryagin's classical construction, the characters of the group \mathfrak{G} form a commutative group into which a locally compact topology is naturally introduced. The group $\widehat{\mathfrak{G}}$ thus obtained is called dual to the group \mathfrak{G} . Pontryagin's duality principle states that the group dual to the group $\widehat{\mathfrak{G}}$ is isomorphic to the group \mathfrak{G} . The first partial generalization of Pontryagin's duality principle to the case of compact (noncommutative) groups was obtained by Tanaka and M. G. Krein. Later this result was substantially supplemented by M. G. Krein, who obtained the duality principle for compact groups^(1,2). Finally, in a recent very interesting paper⁽³⁾ Stinespring, in essence, transferred the first part of M. G. Krein's duality principle⁽²⁾, Theorem 16a) to unimodular locally compact groups. The main difficulty standing in the way of transferring the duality principle to noncommutative groups consists in the fact that the object dual to a group is not a group, but an object, generally speaking, of another nature (a block algebra)⁽²⁾. In this connection it is natural to attempt to generalize the notion of a group so that, in the class of objects thus obtained, duality should hold. The new notion of a group must include, as special cases, locally compact groups and the objects dual to them. In the present note such a generalization of the notion of a group is proposed (Sec. 2). On the objects obtained it is possible to develop harmonic analysis rather far and to establish the duality principle (Secs. 4 and 3). The main idea of the generalization of the notion of a group is as follows. Let \mathfrak{G} be a group. Denote by \mathfrak{M} the commutative ring (with respect to the operation of multiplication) of bounded functions on \mathfrak{G} . The mapping $f(y) \rightarrow f(xy)$ ($f \in \mathfrak{M}$) is an isomorphism of \mathfrak{M} into $\mathfrak{M} \times \mathfrak{M}$. Thus, \mathfrak{G} may be regarded as the commutative ring \mathfrak{M} with a given isomorphism of \mathfrak{M} into $\mathfrak{M} \times \mathfrak{M}$. We obtain the generalization under consideration by discarding in this definition the requirement that the ring \mathfrak{M} be commutative. In view of the applications to harmonic analysis, we leave aside the purely algebraic aspects of this definition and below define the generalization of a unimodular group not abstractly, but rather as a concrete representation. Let us note in conclusion that the articles^(2,3) served as the starting point for the present note.

2. Denote by Φ a certain isomorphism of the ring \mathfrak{M} into its tensor square $\mathfrak{M} \times \mathfrak{M}$. Supplying the rings with indices \mathfrak{M}_α , $\mathfrak{M}_\beta \times \mathfrak{M}_\gamma$, we shall also denote this isomorphism by $\Phi_\alpha^{\beta\gamma}$. Consider now the rings $\mathfrak{M}_\lambda \times \mathfrak{M}_\alpha$ and $\mathfrak{M}_\mu \times \mathfrak{M}_\beta \times \mathfrak{M}_\gamma$ (all \mathfrak{M}_i are identical) and define the mapping $\Phi_{\lambda,\alpha}^{\alpha,\beta\gamma}$, setting $a \times b \rightarrow a \times \Phi_\alpha^{\beta\gamma}(b)$ ($a, b \in \mathfrak{M}$). This mapping-

is one-to-one: it extends to an isomorphism of the first ring onto the second. Analogously one can define the isomorphism $\Phi_{\lambda\alpha}^{\beta\gamma}$.

Let \mathfrak{H} be a Hilbert space, \mathfrak{M} the standard ring of operators in \mathfrak{H} ⁽⁴⁾, and let Φ be a $*$ -isomorphism of the ring \mathfrak{M} into the ring of operators $\mathfrak{M} \times \mathfrak{M}$ (with $\Phi(I) = I \times I$), satisfying the following associativity condition:

$$\alpha) \quad \Phi_{13}^{123} \Phi_3^{13} = \Phi_{23}^{123} \Phi_3^{23}.$$

We shall also assume that an involution $A \rightarrow A^+$ ($A \in \mathfrak{M}$) is defined on \mathfrak{M} , satisfying the conditions

$$\beta) \quad (\lambda A + \mu B)^+ = \bar{\lambda} A^+ + \bar{\mu} B^+, \quad (AB)^+ = A^+ B^+, \quad (A^+)^* = (A^*)^+, \quad (A^+)^+ = A,$$

$$\Phi(A^+) = \tilde{\Phi}(A)^+ \quad (A, B \in \mathfrak{M})$$

(here $A \times B = B \times A$, $(A \times B)^+ = A^+ \times B^+$).

Let m be a regular measure on \mathfrak{M} ⁽⁴⁾ (as in ⁽⁴⁾, m denotes both the measure (gage) and the integral with respect to this measure). Construct, from the measure m , the space $L_1(\mathfrak{M})$ ⁽⁴⁾. Suppose that

$$\gamma) \quad m(A) = \overline{m(A^+)}, \quad m \times m((A \times B)\Phi(T)^*) = \overline{m \times m((T \times B^+)\Phi(A)^*)}$$

$$(A, B \in L_1(\mathfrak{M}) \cap \mathfrak{M}, \quad T \in \mathfrak{M}).$$

The correspondence

$$A \times B \rightarrow V[A \times B] = \Phi(B^*)(A^+ \times I) \quad (A, B \in L_1(\mathfrak{M}) \cap \mathfrak{M}),$$

when extended to all of $L_2(\mathfrak{M})$, defines an antilinear operator V (i.e. $V(\lambda h + \mu h') = \bar{\lambda} Vh + \bar{\mu} Vh'$). We assume that

$$\delta) \quad V^2 = I.$$

The ring \mathfrak{M} , with fixed $*$ -isomorphism Φ , involution, and measure m satisfying conditions $\alpha) - \delta)$, will be called a **ring group**. The ring \mathfrak{M} will be called the **carrier** of the ring group, the measure m an **invariant measure**, and the isomorphism Φ the **defining isomorphism**. A ring group will be called **commutative** if $\tilde{\Phi} = \Phi$. The equality

$$m \times m((A \times B)\Phi(T)) = m((A * B)T)$$

for fixed operators $A, B \in L_1(\mathfrak{M})$ and an arbitrary operator T from \mathfrak{M} uniquely determines an operator $A * B$ from $L_1(\mathfrak{M})$ —the **convolution** of the operators A and B (cf. (3)). From condition $\alpha)$ follows the associativity of the convolution operation.

Theorem 1. *With respect to the convolution operation $*$ and the involution $+$, the operators from $L_1(\mathfrak{M})$ form a normed ring with involution—a group ring (the norm in the ring is the L_1 -norm of the operator).*

As an example we show that a locally compact unimodular group G is a ring group. In this case the space \mathfrak{H} is the space $L_2(G)$, constructed with respect to Haar measure. \mathfrak{M} is the commutative ring of bounded measurable functions on G , regarded as operators in \mathfrak{H} (operators of multiplication by a function). The ring \mathfrak{M} is standard. The defining isomorphism is the mapping

$$A(y) \rightarrow A(xy) \quad (A(x) \in \mathfrak{M}; x, y \in G).$$

Condition $\alpha)$ takes the form

$$A(x(yz)) = A((xy)z).$$

It obviously holds by associativity of the group operation. Define the involution by setting

$$A^+(x) = \overline{A}(x^{-1}).$$

Then the conditions $\beta)$ are fulfilled. Note that

$$A^*(x) = \overline{A}(x).$$

Finally, as the measure m we take Haar measure. Then

$$m(A) = \int A(x) dx.$$

The conditions $\gamma)$ take the form

$$\int A(x) dx = \overline{\int A(x^{-1}) dx}$$

and

$$\iint A(x)B(y)T(xy) dx dy = \overline{\iint T(x)B(y^{-1})A(xy) dx dy},$$

and they hold by virtue of the two-sided invariance of Haar measure. $VF(x, y) = F(x^{-1}, xy)$, whence

$$V^2F(x, y) = F(x, y).$$

Thus G is a ring group. According to the definition, convolution is determined by the equality

$$\iint A(x)B(y)T(xy) dx dy = \int (A * B)(y)T(y) dy,$$

whence, as usual, we obtain

$$(A * B)(y) = \int A(x)B(x^{-1}y) dx.$$

3. Let \mathfrak{H} be a Hilbert space and D its dense subspace, on which two multiplication operations $a \cdot b$ and $a \circ b$ and two involutions a and \bar{a} are defined. We shall say that a **double Hilbert ring** Γ is defined if each of the pairs of operations \cdot , \circ and $\bar{\cdot}$, $\bar{\circ}$,

defines a Hilbert ring ⁽⁴⁾, and in $\mathfrak{H} \times \mathfrak{H}$ there exists a bounded operator W for which

$$(a \times b^\nabla, W(a'^\Delta \times b)) = (aa', b \circ b') \quad (a, a', b, b' \in \mathfrak{D}).$$

One of the pairs of operations, for example \cdot, Δ , shall be called the first, and the other the second. The definition of a double Hilbert ring is symmetric with respect to both pairs of operations. The operators $x \rightarrow a \cdot x$ and $x \rightarrow a \circ x$ ($a \in \mathfrak{H}$) are bounded operators in \mathfrak{H} . We denote their extensions to all of \mathfrak{H} , respectively, by $L_a(\Delta)$ and $L_a(\nabla)$. The Neumann rings generated by the operators $L_a(\Delta)$ ($a \in \mathfrak{D}$) and $L_a(\nabla)$ ($a \in \mathfrak{D}$) will be called, respectively, the **first** and the **second Neumann ring** of the ring Γ . We denote them by \mathfrak{R}_Δ and \mathfrak{R}_∇ .

Theorem 2. *To each ring group \mathfrak{G} there corresponds a double Hilbert ring Γ , one of whose Neumann rings (for example \mathfrak{R}_Δ) coincides with the carrier of the group, and moreover*

a)

$$L_a(\Delta) * L_b(\Delta) = L_{a \circ b}(\Delta), \quad L_a(\Delta)L_b(\Delta) = L_{ab}(\Delta); \quad (a, b \in \mathfrak{D});$$

b)

$$L_a^+(\Delta) = L_{a^\nabla}(\Delta), \quad L_a^*(\Delta) = L_{a^\Delta}(\Delta);$$

c) *the measure m is the canonical measure on \mathfrak{R}_Δ ⁽⁴⁾, i.e.*

$$m(L_a(\Delta)L_b^*(\Delta)) = (a, b).$$

The ring Γ is uniquely determined by the group \mathfrak{G} .

We shall say that the ring Γ and the group \mathfrak{G} are **connected by a pair of operations** (in the present notation, by the first pair). The ring \mathfrak{R}_{∇} will also be called the Neumann ring of the group \mathfrak{G} and will be denoted by \mathfrak{M} .

Theorem 3. *If a double Hilbert ring Γ is connected, by one pair of operations, with a ring group \mathfrak{G} , then it is also connected with some ring group $\widehat{\mathfrak{G}}$ by the second pair of operations. The ring group $\widehat{\mathfrak{G}}$ is uniquely determined by the ring Γ and by the choice of the pair of operations.*

We shall call the group $\widehat{\mathfrak{G}}$ **dual** with respect to the group \mathfrak{G} . From Theorems 2 and 3 it follows immediately that

Theorem 4 (duality principle). *Let \mathfrak{G} be a ring group and $\widehat{\mathfrak{G}}$ the ring group dual to it. The group dual to $\widehat{\mathfrak{G}}$ coincides with \mathfrak{G} .*

4. Consider a double Hilbert ring Γ , connected with a ring group \mathfrak{G} by the first pair of operations and with $\widehat{\mathfrak{G}}$ by the second. Let A be an operator from $\mathfrak{M} = \mathfrak{R}_{\Delta}$, which can be represented in the form $A = L_a(\Delta)$ ($a \in \mathfrak{D}$). Put $\widehat{A} = L_a(\nabla) \in \mathfrak{R}_{\nabla} = \widehat{\mathfrak{M}}$. We shall call the operator \widehat{A} the **Fourier transform** of the operator A . It is obvious that the Fourier transform of the operator \widehat{A} is the operator A . If $A = L_a(\Delta)$ and $B = L_b(\Delta)$, then

$$(\widehat{A * B}) = (L_a(\Delta) * L_b(\Delta)) = \widehat{L_{ab}(\Delta)} = L_{ab}(\nabla) = L_a(\nabla)L_b(\nabla) = \widehat{A} \widehat{B}$$

and

$$\widehat{A^+} = L_a^+(\Delta) = \widehat{L_{a\nabla}(\Delta)} = L_{a\nabla}(\nabla) = L_a^*(\nabla) = \widehat{A^*}.$$

Thus, by virtue of the Fourier transform, the convolution and the involution of operators of the carrier (for which the Fourier transform is defined) are carried respectively into the product and the adjoint of their Fourier transforms.

According to Theorem 2 c), we obtain the Plancherel formula:

$$m(AB^*) = m(L_a(\Delta)L_b^*(\Delta)) = (a, b) = \widehat{m}(L_a(\nabla)L_b^*(\nabla)) = \widehat{m}(\widehat{A} \widehat{B}^*).$$

As usual, it permits one to extend the Fourier transform to all operators from $L_2(\mathfrak{M})$. In this case the Fourier transform carries out an isometric mapping of $L_2(\mathfrak{M})$ onto $L_2(\widehat{\mathfrak{M}})$. Similarly to what holds for ordinary groups, one can extend the definition of the Fourier transform also to all of $L_1(\mathfrak{M})$. The Fourier transforms of operators from $L_1(\mathfrak{M})$ are bounded operators from $\widehat{\mathfrak{M}}$. Let us note

also the equalities $\widehat{\mathfrak{M}} = \mathfrak{N}$, $\widehat{\mathfrak{N}} = \mathfrak{M}$, following from the definitions given above. Thus, in passing to the dual group, the carrier and the Neumann ring exchange roles.

5. In § 2 it was shown that every unimodular group is, in a certain sense, a ring group.

Theorem 5. *A ring group \mathfrak{G} is a locally compact unimodular group if and only if the carrier \mathfrak{G} is commutative.*

It follows easily from Theorem 5 that, in order for a ring group to be dual to a unimodular group, it is necessary and sufficient that it be commutative, or, equivalently, that its Neumann ring be commutative. This assertion, together with Theorem 4, may be regarded as a generalization of M. G. Krein' s duality principle² to unimodular groups.

If \mathfrak{G} is a commutative locally compact group, then it has a commutative carrier and a commutative Neumann ring. The ring group $\widehat{\mathfrak{G}}$ has the same property (see § 4). Therefore $\widehat{\mathfrak{G}}$ is a commutative locally compact group. We have arrived at Pontryagin' s duality principle.

Received
2 XII 1960

CITED LITERATURE

- ¹ M. G. Krein, DAN, **30**, No. 1, 9 (1941).
² M. G. Krein, Ukr. Math. Zh., **2**, No. 1, 10 (1950).
³ W. F. Steinespring, Trans. Am. Math. Soc., **90**, No. 1, 15 (1959).
⁴ J. E. Segal, Ann. Math., **57**, No. 3, 401 (1953).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.