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Abstract

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MATHEMATICS

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ON A CERTAIN MAPPING OF HILBERT SPACE INTO ITSELF

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In the present note we construct a uniformly continuous in both directions homeomorphism of Hilbert space H onto a certain subset of this space, all points of which have nonnegative coordinates less than or equal to one. (Hilbert space is assumed to be given in some orthonormal basis.)

1. Let an orthonormal basis $\{e_1, e_2, \dots, e_r, \dots\}$ be given in the Hilbert space H. Consider the subspace H_2 of the space H having as basis vectors the vectors $\{e_2, e_4, \dots, e_{2n}, \dots\}$, and define the linear mapping $\varphi : H \to H_2$, given on the basis vectors by the formula $\varphi(e_k) = e_{2k}$. We note that any vector of the space $H_2 = \varphi(H)$ can be represented in a unique way as a linear combination with nonnegative coefficients of the vectors $\{e_2, e_4, \dots, e_{2n}, \dots\}$ and $\{-e_2, -e_4, \dots, -e_{2n}, \dots\}$ (under the condition that the vectors e_{2n} and $-e_{2n}$ do not both occur with nonzero coefficients). If in this linear combination, instead of the vectors $-e_{2k}$, we substitute the vectors e_{2k-1}, then we obtain a mapping $\psi : H_2 \to H$, whereby all coordinates of the points (vectors) belonging to the image of the space H_2 are nonnegative. Combining the mappings φ and ψ, we obtain a mapping $\psi\varphi$ of the whole space H into a subset having nonnegative coordinates.

We note that the mutual continuity of the mappings φ and ψ, as well as the uniformity of the homeomorphism φ, are obvious, and we shall prove the uniformity of the homeomorphism ψ.

Indeed, let M and N be two points of H_2, and let the point M have coordinates $(0, x_2, 0, x_4, 0, \dots, x_{2k}, 0, \dots)$, while N has coordinates
$(0, y_2, 0, y_4, 0, \dots, y_{2k}, 0, \dots)$; if for all k the signs of the numbers x_{2k} and y_{2k} coincide*, then the distance between the points M and N and the distance between the points $\psi(M)$ and $\psi(N)$ are the same. Now let $\{x_{2k_1}, x_{2k_2}, \dots, x_{2k_r}, \dots\}$ be the collection of those coordinates of the point M whose sign coincides with the sign of the corresponding coordinates, by number, of the point N. Consider the point P with coordinates

$(0, 0, \dots, 0, x_{2k_1}, 0, \dots, 0, x_{2k_2}, \dots)$, and join the point P with the points M and N. It is clear that the segment MP is perpendicular to the plane spanned by the vectors $e_{2k_1}, e_{2k_2}, e_{2k_3}, \dots, e_{2k_r}, \dots$. Moreover, the triangle MPN has an obtuse angle at the point P, since the scalar product of the vectors $\overrightarrow{PM}$ and $\overrightarrow{PN}$ is negative. Considering the scalar product of the vectors $\overrightarrow{\psi(M)\psi(P)}$ and $\overrightarrow{\psi(N)\psi(P)}$, we see that in the triangle $\psi(M)\psi(P)\psi(N)$ the angle at the point $\psi(P)$ is straight. From what has been said above it is clear that the segment MP is equal to the segment $\psi(M)\psi(P)$, and the segment NP is equal to $\psi(N)\psi(P)$. Denoting the distances MP, NP, MN, and $\psi(M)\psi(N)$, respectively, by α, β, a, and r, and taking into account the remark about the angles of the triangles MNP and $\psi(M)\psi(N)\psi(P)$, we have the relations

$$\max(\alpha, \beta) \leq a \leq \alpha + \beta, \qquad r = \sqrt{\alpha^2 + \beta^2}.$$

* We regard zero as having both the plus sign and the minus sign.

Hence

$$\frac{\max(\alpha, \beta)}{\sqrt{\alpha^2 + \beta^2}} \leq \frac{a}{r} \leq \frac{\alpha + \beta}{\sqrt{\alpha^2 + \beta^2}}$$

or

$$\frac{1}{\sqrt{2}} \leq \frac{a}{r} \leq \sqrt{2}.$$

From what has been said, the desired uniformity follows.

2. **Bending of the space.** Let us perform the mapping $\varphi : H \to H$, constructed in item 1, and, for simplicity, renumber the basis vectors of the space H_2 again consecutively, denoting them by $\mathbf{g}_k$, and also renumber consecutively the basis vectors of the factor space $H_1 = H/H_2$ (the complement H_1 of the space H_2 in H), denoting them by $\mathbf{f}_k$. (The coordinates in H_2 are denoted by x_k, in H_1 by y_k.) If to the basis of the space H_2 we adjoin the vectors $\mathbf{f}_{k_1}, \mathbf{f}_{k_2}, \dots, \mathbf{f}_{k_r}$, then we shall denote the corresponding space by $(H_2, \mathbf{f}_{k_1}, \mathbf{f}_{k_2}, \dots, \mathbf{f}_{k_r})$.

We now define the mapping

$$C^k_{i,n} = C^k_{i,n} : H_2 \to (H_2, \mathbf{f}_k)$$

as follows. All coordinates of the point $C^k_{i,n}(M)$, where $M \in H_2$, except the i-th, coincide with the coordinates of the point M. If the i-th coordinate of the point M is less than n, then the i-th coordinate of the point $C^k_{i,n}(M)$ also remains equal to the corresponding coordinate of the point M, while the coordinate y_k along the vector $\mathbf{f}_k$ is set equal to zero. If, however, the i-th coordinate x_i of the point M is greater than n, then the i-th coordinate of the point $C^k_{i,n}(M)$ is set equal to n, and the coordinate y_k along the vector $\mathbf{f}_k$ equal to $x_i - n$. Figuratively, this mapping may be represented as a bending of the space H_2 along the plane $x_i = n$ at a right angle toward itself in the direction of the vector $\mathbf{f}_k$; therefore we call it a bending $C^k_{i,n}$.

Suppose the space H_2 has already been bent in the directions $\mathbf{f}_{k_1}, \mathbf{f}_{k_2}, \dots, \mathbf{f}_{k_r}$ along $(i_1, n_1), (i_2, n_2), \dots, (i_r, n_r)$, and no k_s and no pairs (i_s, n_s) coincide with one another; then define the bending:

$$C^* = C^k_{i,n} : C^{k_r}_{i_r,n_r} C^{k_{r-1}}_{i_{r-1},n_{r-1}} \cdots C^{k_1}_{i_1,n_1}(H_2) \to (H_2, \mathbf{f}_{k_1}, \dots, \mathbf{f}_{k_{r-1}}, \mathbf{f}_{k_r}, \mathbf{f}_k)$$

(where $k \neq k_s$ and $(i, n) \neq (i_s, n_s)$ for all s) as follows. If $i \neq i_s$ for all s, then the bending C^* is defined as the mapping induced by the bending

$$C^k_{i,n} : (H_2, \mathbf{f}_{k_1}, \dots, \mathbf{f}_{k_r}) \to (H_2, \mathbf{f}_{k_1}, \dots, \mathbf{f}_{k_r}, \mathbf{f}_k)$$

(the coordinates y_s for $1 \leq s \leq r$ are unchanged).

If $i = i_s$ for some marked s, but $n < n_s$ for all marked s, then the mapping C^* is defined as in the preceding case; if, however, there is such a marked s that $n_s < n$ (we assume n_s maximal among those possessing this property), then we define our bending C^* as the induced bending of the space $(H_2, \mathbf{f}_{k_1}, \dots, \mathbf{f}_{k_r})$ in the direction of the vector $\mathbf{f}_k$, but now along the plane $y_{k_s} = n_s - n$, i.e. by the bending $C^k_{k_s, n_s - n}$.

Theorem. The mappings $C^k_{i,n} C^l_{j,m} : H_2 \to (H_2, \mathbf{f}_l, \mathbf{f}_k)$ and $C^l_{j,m} C^k_{i,n} : H_2 \to (H_2, \mathbf{f}_k, \mathbf{f}_l)$ coincide.

Let us note, first, that the spaces $(H_2, \mathbf{f}_k, \mathbf{f}_l)$ and $(H_2, \mathbf{f}_l, \mathbf{f}_k)$ coincide.

Suppose first that $i = j$ and, for definiteness, $m > n$. Take a point M whose i-th coordinate x_i is less than or equal to n; then

$$C^l_{j,m} C^k_{i,n}(M) =$$

$= C^k_{i,n} C^l_{jm}(M) = M$, obviously. Next take a point M for which $m \geqslant x_i > n$; then $C^l_{j,m}(M) = M$ and $C^l_{j,m}(C^k_{i,n}(M)) = C^k_{i,n}(M)$, or, substituting the first equality into the right-hand side of the second, we have $C^l_{j,m}(C^k_{i,n}(M)) = C^k_{i,n}(C^l_{j,m}(M))$, i.e. $C^l_{j,m} C^k_{i,n}(M) = C^k_{i,n} C^l_{j,m}(M)$.

There remains only the case $x_i > m$; in this case the coordinate y_k with respect to the vector $\mathbf{f}_k$ of the point $C^k_{i,n}(M)$ is equal to $x_i - n$, at the point $C^l_{j,m} C^k_{i,n}(M)$ the coordinate y_k is equal to $m - n$, and the coordinate $y_l = x_i - m$; the coordinates of the point $C^k_{i,n}C^l_{j,m}(M)$ are examined in exactly the same way, which gives us the proof of the theorem if $i = j$.

If, however, $i \neq j$, one can consider the very same cases and verify the validity of our theorem. For example, if $x_i > n$ and $x_j > m$, where x_i and x_j are coordinates of the point M, then the i-th coordinates both of the point $C^k_{i,n}C^l_{j,m}(M)$ and of the point $C^l_{j,m}C^k_{i,n}(M)$ are equal to n; the j-th coordinates of the same points are equal to m, respectively, $y_k = x_i - n$, $y_l = x_j - m$, for both the one and the other point, and, since the remaining coordinates of the point M are unchanged under our mappings, it follows that the points $C^l_{j,m}C^k_{i,n}(M)$ and $C^k_{i,n}C^l_{j,m}(M)$ coincide.

Let us note that from our theorem there follows the commutativity of any finite number of bendings.

3. Construction of the mapping C. Number consecutively all pairs (i, n) for $i > 0$, $n > 0$; if the pair (i, n) has received the number $k = k_{i,n}$, then denote the mapping $C^k_{i,n}$ by C^k, and construct the mapping $C = \dots C^k \dots C^2C^1 : H_2 \to (H_2, \mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k, \dots) = H$. Let us record the obvious fact that the mapping C is a homeomorphism. Let us also note that the mapping C carries the whole set of points of the space H_2 having nonnegative coordinates into a subset of the set of those points of the space H whose nonnegative coordinates are less than or equal to one. The latter is easy to see if one takes into account that each point $M \in H_2$ has only a finite number of coordinates greater than one; consequently, only a finite number of the mappings $C^k_{i,n}$ do not leave it fixed, i.e., in order to obtain the point $C(M)$, it is enough to consider only a finite number of mappings C^k, which we may consider in any order. If, for example, $n+1 > x_i \geqslant n$ and x_i is the first coordinate of the point M greater than one, then we first consider the mapping $C' = C^{k_{i,n}}_{i,n}C^{k_{i,n-1}}_{i,n-1} \dots C^{k_{i,1}}_{i,1} : H_2 \to (H_2; \mathbf{f}_{k_{i,1}}, \mathbf{f}_{k_{i,2}}, \dots, \mathbf{f}_{k_{i,n}})$; it is clear that the number of coordinates of the point $C'(M)$ greater than one is one less than for the point M. Similarly we arrive at the fact that all coordinates of the point $C(M)$ are less than or equal to one.

The uniformity in both directions of the homeomorphism $C : H_2 \to H$ is established as in the case of the mapping ψ.

4. It is obvious that the mapping we need is given by the formula $C\varphi\psi\varphi : H \to H$.

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Note: Figure translations are in progress. See original paper for figures.