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E. B. VUL

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Abstract

Full Text

E. B. VUL

ON THE UNIQUENESS OF REPRESENTATION OF CERTAIN CLASSES OF POSITIVE-DEFINITE GENERALIZED FUNCTIONS

(Presented by Academician M. V. Keldysh, 8 VIII 1960)

In the work of I. M. Gel' fand and S. Do-shin ¹, questions connected with representations of positive-definite even linear functionals defined on the spaces Z and Z_2^2 of basic functions were considered. Below, some of these questions are considered for other spaces of basic functions.

Following the book of I. M. Gel' fand and G. E. Shilov ², we introduce the space W_M^Ω of entire functions $\varphi(z) = \varphi(x + iy)$ satisfying the inequalities

$$|\varphi(x + iy)| \leq ce^{-M(a|x|)+\Omega(b|y|)},$$

where the positive numbers a , b , and c depend on the function $\varphi(z)$, and M and Ω are increasing functions having continuous increasing derivatives. Denote by $W_M^{\Omega 0}$ the subspace of the space W_M^Ω consisting of even functions. Conditions for the nontriviality of such spaces were investigated in the works of S. Mandelbrojt ³ and K. I. Babenko ⁴.

Consider on the space $W_M^{\Omega 0}$ a linear functional T , given in the form

$$(T, \varphi) = \int_0^\infty \int_{-\infty}^\infty \cos x\lambda \varphi(x) dx d\mu(\lambda) + \int_0^\infty \int_{-\infty}^\infty \operatorname{ch} x\lambda \varphi(x) dx d\sigma(\lambda), \quad (1)$$

where the measures $\sigma(\lambda)$ and $\mu(\lambda)$ are such that

$$\int_0^\infty \left| \int_{-\infty}^\infty \cos x\lambda \varphi(x) dx \right| d\mu(\lambda) < \infty,$$

$$\int_0^\infty \left| \int_{-\infty}^\infty \operatorname{ch} x\lambda \varphi(x) dx \right| d\sigma(\lambda) < \infty$$

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for all functions $\varphi \in \overset{0}{W}_M^\Omega$. It is easy to see that the functional T satisfies the relation $T(\varphi * \varphi^*) \geq 0$, where $\varphi \in \overset{0}{W}_M^\Omega$, $\varphi^*(z) = \varphi(-\bar{z})$, and is therefore positive-definite. Functionals of this type were studied in the work of A. G. Kostyuchenko and B. S. Mityagin ⁵. We shall be interested in the question of when, in the expansion (1), the measures $\mu(\lambda)$ and $\sigma(\lambda)$ are determined uniquely.

Let us pass to the Fourier transform of the space $\overset{0}{W}_M^\Omega$. It is shown in ⁶ that the Fourier transform of the space $\overset{0}{W}_M^\Omega$ is the space

$W_{\Omega_1}^{M_1}$, where Ω_1 is the function conjugate in the sense of Young* to M , and M_1 is the function conjugate in the sense of Young to Ω . It is obvious that the Fourier transform of the space $\overset{0}{W}_M^\Omega$ will be the space $\overset{0}{W}_{\Omega_1}^{M_1}$. The linear functional T , defined on the space $\overset{0}{W}_M^\Omega$ by formula (1), is transformed into the linear functional T^* on the space $\overset{0}{W}_{\Omega_1}^{M_1}$, written in the form

$$(T^*, \psi) = \int_0^\infty \psi(x) d\mu(x) + \int_0^\infty \psi(iy) d\sigma(y), \quad (2)$$

where

$$\int_0^\infty |\psi(x)| d\mu(x) < \infty, \quad \int_0^\infty |\psi(iy)| d\sigma(y) < \infty.$$

The question of the uniqueness of the representation of the functional T in the form (1) has thereby been reduced to the question of the uniqueness of the representation of the functional T^* in the form (2). Under the assumption that the finite limit $\lim_{x \rightarrow \infty} \frac{xM_1'}{M_1}$ exists, Theorems 1 and 2 hold.

Theorem 1. If

$$\int_1^\infty \frac{M_1(x)}{x^3} dx = \infty,$$

then to every positive-definite functional T on the space $\overset{0}{W}_M^\Omega$, given in the form (1), there correspond unique measures $\mu(\lambda)$ and $\sigma(\lambda)$.

Theorem 2. If

$$\int_1^\infty \frac{M_1(x)}{x^3} < \infty,$$

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then there exist measures $\mu(\lambda) \neq \mu_1(\lambda)$ and $\sigma(\lambda) \neq \sigma_1(\lambda)$ such that, for any function $\psi(z) \in W_{\Omega_1}^{M_1}$,

$$\int_0^\infty \psi(x) d\mu(x) + \int_0^\infty \psi(iy) d\sigma(y) = \int_0^\infty \psi(x) d\mu_1(x) + \int_0^\infty \psi(iy) d\sigma_1(y) \quad (3)$$

and the integrals in (3) converge absolutely.

Proof of Theorem 1. It suffices to consider only the spaces $W_{M_1}^{M_1}$, since from the condition that the space is nonempty it follows that $W_{M_1}^{M_1} \in W_{\Omega_1}^{M_1}$ (4). Suppose that the representation of the functional T^* in the form (2) is not unique. Then there exist such absolutely additive set functions $\tilde{\mu}$ and $\tilde{\sigma}$ that

$$\int_0^\infty \psi(x) d\tilde{\mu}(x) + \int_0^\infty \psi(iy) d\tilde{\sigma}(y) = 0, \quad (4)$$

where $\psi(z)$ is an arbitrary function from $W_{M_1}^{M_1}$, and $\tilde{\mu}$ and $\tilde{\sigma}$ have bounded variation on every finite interval and the integral in (4) converges absolutely.

To each function $\psi(z) \in W_{M_1}^{M_1}$ assign the function

* If $g(x)$ is the inverse function to $\Omega'(x)$, then the function $\int_0^x g(t) dt$ is called the function conjugate in the sense of Young to the function $\Omega(x)$.

$\psi_1(z) = \psi(\sqrt{z})$. Then for the function $\psi_1(z)$ the inequalities

$$\begin{aligned} |\psi_1(x)| &\leq ce^{-M_1(a\sqrt{x})}, & x > 0; \\ |\psi_1(x)| &\leq c'e^{M_1(b\sqrt{|x|})}, & x < 0. \end{aligned} \quad (5)$$

hold.

It follows from (4) that, for any entire function $\psi_1(z)$ satisfying the inequalities (5),

$$\int_{-\infty}^\infty \psi_1(x) d\nu(x) = 0, \quad (6)$$

where

$$d\nu(x) = \begin{cases} d\tilde{\mu}(x^2), & x \geq 0; \\ d\tilde{\sigma}(x^2), & x < 0. \end{cases}$$

Moreover, the integral in (6) converges absolutely for any function $\psi_1(x)$, if $\psi_1(z^2) \in \overset{0}{W}_{M_1}^{M_1}$.

It can be shown that for any $c > 0$ there exists a function $\varphi(z)$ from $\overset{0}{W}_{M_1}^{M_1}$ satisfying the inequality

$$|\varphi(iy)| > e^{cM_1(|y|)}. \quad (7)$$

It then follows from (6) that

$$\int_{-\infty}^0 e^{cM_1(\sqrt{|y|})} d\nu(y) < \infty$$

for any $c > 0$. Put $d\nu_1(x) = \psi_1^2(x)d\nu(x)$, where $\psi_1(z^2) \in \overset{0}{W}_{M_1}^{M_1}$. If $P(x)$ is a polynomial, then $P(x)\psi_1^2(x)$ again satisfies the inequalities (5), and

$$\int_{-\infty}^{\infty} P(x) d\nu_1(x) = 0. \quad (8)$$

We shall now show that it follows from the hypothesis of the theorem that

$$\int_{-\infty}^{\infty} f(x) d\nu_1(x) = 0, \quad (9)$$

where $f(x)$ is an arbitrary continuous finite function. Indeed, using S. N. Bernstein's theorem (see the survey article ⁶), for any $\varepsilon > 0$ we find a polynomial $P(x)$ such that for all x

$$|P(x) - f(x)|e^{-M_1(\sqrt{|x|})} < \varepsilon.$$

Then

$$\left| \int_{-\infty}^{\infty} f(x) d\nu_1(x) \right| = \left| \int_{-\infty}^{\infty} (f(x) - P(x)) d\nu_1(x) \right| \leq \varepsilon \int_{-\infty}^{\infty} e^{M_1(\sqrt{|x|})} d\nu_1(x) = \varepsilon \cdot \text{const.}$$

In view of the arbitrariness of ε , we have $\int_{-\infty}^{\infty} f(x) d\nu_1(x) = 0$. Consequently, $d\nu_1 = d\nu = 0$. The theorem is proved.

Proof of Theorem 2. Let the function $\widetilde{M}(x)$ be such that

$$\lim_{x \rightarrow \infty} \frac{\widetilde{M}(x)}{M(x)} = \infty, \quad \int_1^{\infty} \frac{\widetilde{M}(x)}{x^3} dx = 2 \int_1^{\infty} \frac{\widetilde{M}(\sqrt{x})}{x^2} dx < \infty.$$

In (7), for a nondecreasing function \widetilde{M} for which

$$\int_1^{\infty} \frac{\widetilde{M}(\sqrt{x})}{x^2} dx < \infty,$$

an entire function $F(z)$ is constructed which takes real values on the real axis and satisfies, for all z , the inequality

$$|F(z)| \leq ce^{\beta|y| - \widetilde{M}(\sqrt{|z|})}, \quad \beta < 0.$$

Consider the function $F_1(z) = F(z)e^{i\beta z}$. In the upper half-plane it satisfies the inequality

$$|F_1(z)| < ce^{-\widetilde{M}(\sqrt{|z|})},$$

and therefore, for any function $\psi_1(z), \psi_1(z^2) \in W_{\Omega}^{M_1}$,

$$\lim_{R \rightarrow \infty} \int_{C'_R} \psi(z) F_1(z) dz = 0$$

(where C_R is the semicircle $\text{Im } z \geq 0, |z| = R$), whence it follows that

$$\int_{-\infty}^{\infty} \psi_1(x) F_1(x) dx = 0. \quad (10)$$

Carrying out analogous arguments for the function $F_2(z) = F(z)e^{-i\beta z}$ in the lower half-plane, we obtain that

$$\int_{-\infty}^{\infty} \psi_1(x) F_2(x) dx = 0.$$

Adding (10) and (11), we have

$$\int_{-\infty}^{\infty} \psi_1(\lambda) F(x) \cos \beta x dx = 0.$$

Let

$$dv_1 = \max(0, F(x) \cos \beta x) dx,$$

$$dv_2 = \max(0, -F(x) \cos \beta x) dx.$$

Then

$$\int_{-\infty}^{\infty} \psi_1(x) dv_1(x) = \int_{-\infty}^{\infty} \psi_1(x) dv_2(x)$$

and the measures $\sigma(x^2)$, $\mu(x^2)$ and $\sigma_1(x^2)$, $\mu_1(x^2)$, constructed from the measures ν_1 and ν_2 , define one and the same functional T .

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Note: Figure translations are in progress. See original paper for figures.

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