# MATHEMATICAL PHYSICS

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**Abstract**

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# ON THE INTEGRAL REPRESENTATION OF JOST–LEHMANN–DYSON

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**1.** In the works [1,2] of Jost, Lehmann, and Dyson an integral representation of the causal commutator was obtained. More precisely, the following theorem was proved. Let the generalized function $f(x)$, $x = (x_0, x_1, x_2, x_3) = (x_0, \bar{x})$, belong to the Sobolev-Schwartz space $S^*$ and vanish at spacelike points: $x^2 = x_0^2 - \bar{x}^2 < 0$. Suppose, further, that its Fourier transform $\tilde{f}(p)$ vanishes in a domain $G$, whose boundary consists of two spacelike surfaces*. Then $\tilde{f}(p)$ can be represented in the form

$$\tilde{f}(p) = \int \operatorname{sgn}(p_0 - u_0)\, \delta[(p-u)^2 - \lambda] \psi(u, \lambda)\, du\, d\lambda, \tag{1}$$

where the support of the (generalized) function $\psi(u, \lambda)$ consists of those $u$ and $\lambda \geq 0$ for which the interior $(p-u)^2 > \lambda$ of the two-sheeted hyperboloid $(p-u)^2 = \lambda$ lies entirely outside the domain $G$**.

In this connection Dyson put forward the conjecture (see [2], note 7) that for any open set $G$ the support of the weight function $\psi$ in the representation (1) consists of those $u$ and $\lambda \geq 0$ for which the hyperboloid $(p-u)^2 = \lambda$ lies wholly outside $G$. This conjecture was in fact established by them for such open sets $G = \bigcup G_j$, for which each connected component $G_j$ is bounded by two spacelike surfaces and any points of distinct connected components $G_j$ are separated by spacelike intervals, i.e. $(x_1 - x_2)^2 < 0$, if $x_1 \in G_{j_1}$, $x_2 \in G_{j_2}$, $j_1 \neq j_2$.

**2.** In works [3,4], one of the authors (V. S. V.) established the validity of Dyson' s conjecture for a broader class of domains, namely for such domains $G$ for which the least convex hull $B_0(G)$ with respect to timelike curves is bounded by two spacelike surfaces***, and also for such open sets $G = \bigcup G_j$ ($G_j$ are the connected components of $G$) for which any points of distinct connected components of the set $\bigcup B_0(G_j)$ are separated by spacelike intervals. This result is a consequence of the following theorem [3]: if $f(x) \in S^*$, $f(x) = 0$ for $x^2 < 0$, and $\tilde{f}(p) = 0$ for $p \in G$, where $G$ is an arbitrary domain, then $\tilde{f}(p) = 0$ for $p \in B_0(G)$, where $B_0(G)$ is the least convex hull of $G$ with respect to timelike curves (the definition of $B_0(G)$ is given in [4]).

**3.** The aim of the present note is to clarify the validity of Dyson' s conjecture for domains and open sets $G$ that do not satisfy

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* For notation and definitions, see [4].

** Such hyperboloids were called admissible by Dyson.

*** In [3] this condition is omitted.

restrictions of item 2. It turns out that for them Dyson' s conjecture is then not always valid. For example, let us take as the domain $G$ the half-space $p_0 < 0$. If Dyson' s conjecture were true, then, by virtue of (1), $\tilde{f}(p)$ would be equal to zero in the whole space $R^4$. But a simple example shows that this is not so. Indeed, let

$$\tilde{f}(p) = P(p) \lim_{\varepsilon \to +0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t - p_0)^2 + \varepsilon^2}, \tag{2}$$

where $P(p)$ is an arbitrary polynomial and $\sigma(t) \not\equiv \text{const}$ is a function of bounded variation, constant for $t < 0$. The limit in (2) exists in the sense of $S^*$. The function $\tilde{f}(p)$ constructed by formula (2) is representable as the difference of the boundary values $F(p_0 \pm i0, \bar{p})$ of the function

$$F(\zeta) = P(\zeta) \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t - \zeta_0}, \qquad \zeta = (\zeta_0, \bar{\zeta}) = p + iq, \tag{3}$$

holomorphic in the domain bounded by the analytic hyperplane $\zeta_0 = \rho,\ \rho \geqslant 0$ (in the space $C^4$). Therefore the Fourier transform $f(x)$ of the function $\tilde{f}(p)$ vanishes for $x^2 < 0$ (see ([3,4])).

4. As a second example, let us consider the open set $G$ consisting of two strips:

$$G = [p:\ -n < p_0 < -m \text{ and } m < p_0 < n,\ \bar{p} \text{ arbitrary}]. \tag{4}$$

If Dyson' s conjecture is accepted, then $\tilde{f}(p)$ must be zero in the entire strip $|p_0| < n$. But, taking in (2) $\sigma(t)$ constant on the intervals $(-n, -m)$, $(m, n)$ and nonconstant on the interval $(-n, n)$, we obtain, as above, a counterexample.

5. Denote by $T$ the tubular domain in $C^4$ of the form

$$T = [\zeta = p + iq:\ q^2 > 0,\ \bar{p} \text{ arbitrary}].$$

Let $G$ be an arbitrary open set in $R^4$; by $\tilde{G}$ denote a complex neighborhood of $G$ in $C^4$ such that, if a ball of radius $r$ belongs to $G$, then the corresponding complex ball with the same center and radius $\delta r$ belongs to $\tilde{G}$ ($\delta = 0,1$).

Let $f(x) = 0$ for $x^2 < 0$ and $\tilde{f}(p) = 0$ in the open set $G$. Then, as is known ([4]), there exists a function $F(\zeta)$, holomorphic in the domain $T \cup \tilde{G}$, such that, in the sense of convergence in $S^*$, the following limiting relation holds:

$$\tilde{f}(p) = \lim_{\varepsilon \to +0} \left[ F(p_0 + i\varepsilon, \bar{p}) - F(p_0 - i\varepsilon, \bar{p}) \right]. \tag{5}$$

This result is a consequence of the edge-of-the-wedge theorem ([5,6]).

6. In connection with the result stated in item 5, the problem arises of constructing the envelope of holomorphy $H(T \cup \tilde{G})$ of the domain $T \cup \tilde{G}$. The integral representation (1) makes it possible to do this, at least for those $G$ that satisfy the conditions of item 2 (see ([4])). If these conditions are not fulfilled, however, then it is sometimes possible to construct or estimate $H(T \cup \tilde{G})$. Thus, in the case of the half-space $p_0 < 0$ (see item 3), $H(T \cup \tilde{G})$ is the limit, as $N \to +\infty$, of the sequence $H(T \cup \tilde{G}_N)$, where $G_N$ is the strip $-N < p_0 < 0$. Since for a strip the envelope of holomorphy is known ([5,6]):

$$H(T \cup \tilde{G}_N) = \{ |\bar{q}| < |\operatorname{Im} \sqrt{\zeta_0(\zeta_0 + N)}| \}, \tag{6}$$

then, passing in (6) to the limit as $N \to +\infty$, we find that the desired $H(T \cup \tilde{G})$ is the domain in $C^4$ bounded by the hyperplane $\zeta_0 = \rho, \ \rho \geqslant 0$.

7. Let $G$ consist of the two strips (4). Using the definition of $\widetilde{G}$ (item 5), it is not hard to show that $T \cup \widetilde{G}$ contains a semitubular domain of the form

$$[\zeta = (\zeta_0, \bar{\xi}) : \zeta_0 \in B, \ |\bar{q}| < V(\zeta_0), \ \bar{p} \text{ arbitrary}], \tag{7}$$

where $B$ is the plane of the complex variable $\zeta_0 = p_0 + iq_0$ with three cuts removed:

$$-\infty < p_0 \leq -n, \qquad -m \leq p_0 \leq m, \qquad n \leq p_0 < \infty, \qquad q_0 = 0; \tag{8}$$

$$V(\zeta_0) = \sup_{\zeta_0 \in B} \left\{ |q_0|, \ \frac{\delta}{\sqrt{2}} \left( \frac{n-m}{2} - \left| p_0 - \frac{n+m}{2} \right| \right) \theta \left[ \frac{\delta}{\sqrt{2}} \left( \frac{n-m}{2} - \left| p_0 - \frac{n+m}{2} \right| \right) - |q_0| \right], \right.$$
$$\left. \frac{\delta}{\sqrt{2}} \left( \frac{n-m}{2} - \left| p_0 + \frac{n+m}{2} \right| \right) \theta \left[ \frac{\delta}{\sqrt{2}} \left( \frac{n-m}{2} - \left| p_0 + \frac{n+m}{2} \right| \right) - |q_0| \right] \right\}, \tag{9}$$

where $\theta(\xi) = 1$ for $\xi \geq 0$ and $\theta(\xi) = 0$ for $\xi < 0$.

Taking into account the fact that automorphisms of a domain are preserved also in its envelope of holomorphy (see ([7]), p. 260), and using Bremermann' s theorem ([8]), we conclude that the envelope of holomorphy of the domain (6) has the same form as the domain itself, with $V(\zeta_0)$ necessarily replaced by the least superharmonic majorant $V_0(\zeta_0)$ of the function $V(\zeta_0)$.

8. We shall show that the function

$$V_1(\zeta_0) = \left|\operatorname{Im}\sqrt{\zeta_0^2 - n^2}\right| - \frac{1}{\pi}\int_{-m}^{m} \frac{|q_0|\sqrt{n^2 - t^2}\,dt}{(p_0 - t)^2 + q_0^2} \tag{10}$$

is a superharmonic majorant of the function $V(\zeta_0)$ in the domain $B$. Indeed, the function $\operatorname{Im}\sqrt{\zeta_0^2 - n^2}$ is harmonic and preserves its sign in $B$. Therefore the function $\left|\operatorname{Im}\sqrt{\zeta_0^2 - n^2}\right|$ is also harmonic in $B$. Noting that the modulus of a harmonic function is a subharmonic function ([9]), and taking into account the equality

$$\int_{-m}^{m} \frac{|q_0|\sqrt{n^2 - t^2}}{(p_0 - t)^2 + q_0^2}\,dt = \left|\operatorname{Im}\int_{-m}^{m} \frac{\sqrt{n^2 - t^2}\,dt}{t - \zeta_0}\right|,$$

we conclude that the second term in (10), and consequently $V_1(\zeta_0)$, are superharmonic functions. It remains to prove that $V_1(\zeta_0)$ majorizes the function $V(\zeta_0)$. In view of (9), for this it is sufficient to prove two inequalities:

$$V_1(\zeta_0) - |q_0| \geq 0 \quad \text{for } \zeta_0 \in B, \tag{11}$$

$$V_2(\zeta_0) = V_1(\zeta_0) - \frac{\delta}{\sqrt{2}}\left(\frac{n-m}{2} - \left|p_0 - \frac{n+m}{2}\right|\right) \geq 0 \tag{12}$$

in the triangles

$$\begin{aligned} &m \leq p_0 \leq \frac{n+m}{2}, \qquad 0 \leq q_0 \leq \frac{\delta}{\sqrt{2}}(p_0 - m);\\ &\frac{n+m}{2} \leq p_0 \leq n, \qquad 0 \leq q_0 \leq \frac{\delta}{\sqrt{2}}(n - p_0). \end{aligned} \tag{13}$$

Since the function $V_1(\zeta_0) - |q_0|$ is superharmonic in $B$, in order to prove inequality (11) it is sufficient to prove that $V_1(\zeta_0) = |q_0|$ on the boundary of the domain $B$, i.e. on the cuts (8) and at infinity. But this is easily verified directly (on the cut $|p_0| \leq m$ it is necessary to use the property of the $\delta$-shaped sequence; at the points $\zeta_0 = \pm m$ the function $V_1(\zeta_0)$ has no limiting values). To prove inequality (12), note that the function $V_2(\zeta_0)$ is harmonic inside the triangles (13). Therefore it suffices to establish the inequality $V_2(\zeta_0) \geq 0$ on the boundaries of these triangles:

$$m \leq p_0 \leq n, \quad q_0 = 0 \quad (a); \qquad m \leq p_0 \leq \frac{n+m}{2}, \quad q_0 = \frac{\delta}{\sqrt{2}}(p_0 - m) \quad (b);$$

$$\frac{n+m}{2} \leq p_0 \leq n, \quad q_0 = \frac{\delta}{\sqrt{2}}(n - p_0) \quad (c);$$

$$0 \leq q_0 \leq \frac{\delta}{2\sqrt{2}}(n-m), \quad p_0 = \frac{n+m}{2} \quad (d).$$

On the line (a) inequality (12) reduces to

$$\sqrt{n^2 - p_0^2} \geq \frac{\delta}{\sqrt{2}}\left(\frac{n-m}{2} - \left|p_0 - \frac{n+m}{2}\right|\right), \qquad m \leq p_0 \leq n,$$

which is verified directly. On the lines (b), (c), and (d), inequalities (12) reduce respectively to the inequalities

$$V_1\left(\frac{\sqrt{2}}{\delta}q_0 + m + iq_0\right) \geq q_0, \qquad V_1\left(-\frac{\sqrt{2}}{\delta}q_0 + n + iq_0\right) \geq q_0,$$

$$V_1\left(\frac{n+m}{2} + iq_0\right) \geq q_0,$$

which are ensured by (11).

9. We have not succeeded in showing that the constructed function $V_1(\zeta_0)$ is the least superharmonic majorant of the function $V(\zeta_0)$. Therefore one can only conclude that the envelope of holomorphy of the domain (7) is contained in the semitubular domain of holomorphy

$$|\bar{q}| < V_1(\zeta_0). \tag{14}$$

Since the domain $T \cup \widetilde{G}$ is contained in the domain (14), (14) majorizes the envelope of holomorphy $H(T \cup \widetilde{G})$, i.e.,

$$H(T \cup \widetilde{G}) \subset \left\{ |\bar{q}| < \left|\operatorname{Im}\sqrt{\zeta_0^2 - n^2}\right| - \frac{1}{\pi}\int_{-m}^{m} \frac{|q_0|\sqrt{n^2 - t^2}}{(p_0 - t)^2 + q_0^2}\, dt \right\}. \tag{15}$$

As a lower estimate one may take the sum of the domains of holomorphy of the separate strips [5,6]:

$$\text{either} \quad |\bar{q}| < \left|\operatorname{Im}\sqrt{(\zeta_0 - n)(\zeta_0 - m)}\right|, \qquad \text{or} \quad |\bar{q}| < \left|\operatorname{Im}\sqrt{(\zeta_0 + n)(\zeta_0 + m)}\right|.$$

**Remark.** From the inclusion (15) it follows, in accordance with the example of Sec. 4, that the points

$$[\zeta: \ |p_0| \leq m, \ q = 0, \ \bar{p} \text{ arbitrary}]$$

do not belong to $H(T \cup \widetilde{G})$.

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*Note: Figure translations are in progress. See original paper for figures.*