ON A BOUNDARY-VALUE PROBLEM FOR THE BLASIUS EQUATION

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Abstract

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ON A BOUNDARY-VALUE PROBLEM FOR THE BLASIUS EQUATION

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In the present paper the existence of a solution of the following boundary-value problem will be proved:

$$y''' + 2yy' = 0; \tag{1}$$

$$y(\pm 0) = 0, \quad y'(+0) = y'(-0), \quad y''(+0) = cy''(-0), \quad \lim_{x\to+\infty} y'(x) = a,$$

$$\lim_{x\to-\infty} y'(x) = b \quad (c > 0,\ a \geq 0,\ b \geq 0,\ a \neq b). \tag{1'}$$

The boundary-value problem (1), (1′) for $a = 1,\ b = 1 + \lambda\ (-1 \leq \lambda \leq 1),\ c = 1$ arises in the problem of the mixing of two gas jets ([1]).

Lemma 1. *For any solution of equation (1), one of the following two alternatives holds: either* $y''(x) \neq 0\ (-\infty < x < \infty)$*, or* $y''(x) \equiv 0$.

The proof follows easily from the uniqueness of the solution of equation (1) with initial conditions $y(x_0) = y_0,\ y'(x_0) = y'_0,\ y''(x_0) = 0$.

Theorem 1. *If* $y'(x) > 0\ (-\infty < x < \infty)$*, then both limits* $\lim_{x\to+\infty} y'(x)$*,* $\lim_{x\to-\infty} y'(x)$ *exist and are finite.*

Proof. We shall prove the theorem by contradiction. Since, under the substitution x by $-x$, y by $-y$, equation (1) does not change its form, it is sufficient in Theorems 1 and 2 to consider the case $x \geq 0$. Integrating equation (1) twice, we obtain

$$y'(x) = y'(0) + y''(0)x + \frac{1}{2}\left[\int_0^x y'(x)\,dx\right]^2 - \int_0^x \int_0^t {y'}^2(x)\,dx\,dt,$$

whence it is seen that either $x \to c,\ y \to +\infty,\ y' \to +\infty$, or $y \to +\infty,\ x \to +\infty,\ y' \to +\infty$. Make the change of variables $y' = p(y)$. Then (1) becomes the equation

$$p\frac{d^2p}{dy^2}+\left(\frac{dp}{dy}\right)^2+2y\frac{dp}{dy}=0 \qquad (2)$$

and $\lim_{y\to\infty} p(y)=\infty$. It is easy to see that d^2p/dy^2, beginning with some y, is monotone, since if $d^3p/dy^3=0$, then, differentiating (2), we obtain

$$\frac{d^2p}{dy^2}=\frac{2\,dp/dy}{3\,dp/dy+2y}=-p\left[\left(\frac{dp}{dy}\right)^2+2\frac{dp}{dy}\right],\quad \frac{p}{y^2}=\left(\frac{(\partial p/\partial y)}{y}+2\right)\left(\frac{(3dp/dy)}{y}+2\right). \qquad (3)$$

But $dp/dy>0$, hence $d^2p/dy^2<0$ and $dp/dy=o(y)$, $p=o(y^2)$, and equality (3) is impossible for sufficiently large $y>0$. From the preceding considerations it follows that

$$\lim_{y\to\infty}\frac{d^2p}{dy^2}=0.$$

In this case either $dp/dy\to c>0$,

or $\lim\limits_{y\to\infty} dp/dy=0$. The equality $dp/dy\to c>0$ is impossible, since in equation (2) the last term would be of higher order than the others. Thus, finally,

$$\frac{d^2p}{dy^2}\to 0,\qquad \frac{dp}{dy}\to 0,\qquad p\to\infty,\qquad \text{if } y\to\infty. \qquad (4)$$

Make the change of variables $\rho=uy^2$, $y=e^t$. Equation (2) becomes the equation

$$u\left[u''+3u'+2u\right]+\left[u'+2u\right]^2+u'+2u=0. \qquad (5)$$

From (4) we have, as $t\to\infty$: $du/dt=e^{tdu}/dy=(yp'-2p)/y^2\to 0$, $u\to 0$, $d^2u/dt^2\to 0$. But then from (5) it follows that $du/dt\sim -2u$, $u\sim ce^{-2t}$, $du/dt\to -2ce^{-2t}$ and $du/dy\sim c_1/y^3$. Taking into account that $\lim\limits_{t\to\infty} u=0$, we obtain $u\sim c_1/y^2$ and $p\sim c_2$.

The contradiction obtained proves the theorem.

Theorem 2. Let $y(x)$ be a solution of equation (1) and $\lim\limits_{x\to+\infty} y'(x)=a>0$, $\lim\limits_{x\to-\infty} y'(x)=b>0$, $y(0)=c_0$, $y'(0)=c_1$, $y''(0)=c_2$. Then for every $\varepsilon>0$ one can indicate a $\delta>0$ such that as soon as $|y_0-c_0|<\delta$, $|y_0'-c_1|<\delta$, $|y_0''-c_2|<\delta$, then

$$\max_{-\infty<x<+\infty} |\, y'(x) - y'(x, y_0, y_0', y_0'')\,| < \varepsilon.$$

Proof. Pass to equation (2). Let $\varepsilon > 0$ be given. Make the substitution $p^2 = u$. Then $u(x)$ satisfies the equation

$$\frac{\sqrt{u}}{2}\frac{d^2u}{dy^2} + y\frac{du}{dy} = 0. \tag{6}$$

Denote by $\bar{y}$ such a y that

$$e^{4y^2/a}\int_y^\infty e^{-4y^2/a}\,dy < \max\left\{\frac{\varepsilon}{4}, \frac{a}{4}\right\}, \qquad \frac{du}{dy} < \frac{1}{2} \quad (y > \bar{y}), \qquad \frac{a}{2} < \sqrt{u(\bar{y})}, \tag{7}$$

and by G a neighborhood of the point $\bar{y}$ in which (7) is true and $\sqrt{u(y)} > a/4$; write (6) in the form

$$-\frac{d^2u/dy^2}{du/dy} = \frac{2y}{\sqrt{u}}.$$

If $y \in G$, then

$$-\frac{d^2u/dy^2}{du/dy} < \frac{8y}{a}.$$

Integrating twice, we obtain

$$|\Delta u| < \left|\frac{du}{dy}\right|_{y=\bar{y}} e^{4\bar{y}^2/a}\int_{\bar{y}}^{y} e^{-4y^2/a}\,dy. \tag{8}$$

From inequalities (7) and (8) it follows that, first, $|\Delta u| < a/4$, i.e. $G \supset (\bar{y}, \infty)$, and also $|\Delta u| < \varepsilon/4$ $(\bar{y} < y < \infty)$. Since on a finite segment the solution is a continuous function of the initial data, the theorem is proved.

Theorem 3. Denote $y'(0) = \alpha,\ y''(0) = \beta$. For any $a \geq 0$ and any $\alpha \geq 0$ there exists such a $\beta(\alpha)$ that $y(x)$ satisfies equation (1) and $\lim\limits_{x\to+\infty} y'(x) = a$. Moreover: if $a - \alpha \geq 0$, then $\beta(\alpha)/\alpha$ is a continuous monotonically decreasing function of α; if $a - \alpha < 0$, then, defining $\bar{\beta}(\alpha)$ as the exact upper bound of those $y''(0)$ for which $\lim\limits_{x\to\infty} y'(x) + a$, we obtain that $\bar{\beta}(\alpha)\alpha^{-1}$ is a nonincreasing function of α.

The proof of Theorem 3 in the case $a - \alpha \geq 0$ follows easily from Theorems 1, 2 and from the strict monotonicity, proved in the works of Iglisch (2, 3), of $\lim\limits_{x\to+\infty} y'(x)$ with respect to $y'(0)$ for fixed $y''(0) > 0$. In the case $a - \alpha < 0$, Iglisch’ s arguments, in which the positivity of $y''(x)$ is essentially used, do not go through, since for $a - \alpha < 0$ we have $y''(x) < 0$. Let us make

make the change of variables $y' = p(y)$ and then $p = y^2/u,\ y = e^t$. For $u(x)$ we obtain the equation

$$u\left(2u + \frac{du}{dt} - \frac{d^2u}{dt^2}\right) - 2\frac{du}{dt}\left(2u - \frac{du}{dt}\right) + \left(2u - \frac{du}{dt}\right)^2 + 2u^2\left(2u - \frac{du}{dt}\right) = 0.$$

If $du/dt = 0$, then $d^2u/dt^2 = 6u + 4u^2$, i.e. $du/dt = 0$ no more than once, but $\lim_{t\to-\infty} u = 0,\ \lim_{t\to+\infty} u = \infty,\ u > 0$; therefore $du/dy > 0$ and $du/dy = z(u)$. We obtain the equation

$$\frac{dz}{du} = 6\frac{u}{z} + 3\frac{z}{u} + 4\frac{u^2}{z} - 2u - 7. \tag{9}$$

Since for $u \neq 0$ we have $z \neq 0$, and for $u > 0$ the uniqueness theorem for solutions of equation (9) is valid, if for some $u > 0$ the inequality $z_1(u) > z_2(u)$ holds, then this inequality is true for all $u > 0$. Substituting in (9) $z = y\,du/dy = (2y^2p - yp')/p^2$ and expanding the resulting fractions in a Taylor series, we obtain

$$\frac{dz}{du} = 2 - \frac{3}{2}\frac{p'(0)}{p(0)}y + O(y^2) = 2 - \frac{3}{2}\frac{p'(0)}{\sqrt{p(0)}}\sqrt{u} + o(\sqrt{u}).$$

Let now $p_1'(0) = p_2'(0) < 0,\ p_1(0) > p_2(0)$. Then

$$z_1(0) = z_2(0), \qquad \frac{dz_1}{du} < \frac{dz_2}{du}, \tag{10}$$

if u is small. From the fact that $p_1(0) > p_2(0)$ and from the equality $p = y^2/u$, for sufficiently small $y > 0$ we have $u_1(y) < u_2(y)$. Let $y_0 > 0$ be the first point at which $u_1 = u_2$. Since $u_1(y) < u_2(y)$ if $0 < y < y_0$, then $du_1/dy\big|_{y=y_0} > du_2/dy\big|_{y=y_0}$, i.e. $z_1(u_0) > z_2(u')$, and this entails $z_1(u) > z_2(u)$ for all $u > 0$, which contradicts (10). The case $p_1(0) = p_2(0),\ 0 > dp_1/dy\big|_{y=0} > dp_2/dy\big|_{y=0}$ is considered analogously.

We shall show that $\bar{\beta}(\alpha)\alpha^{-1}$ does not increase when $a = \alpha < 0$. Suppose this is not so and $\alpha_1 > \alpha_2,\ \alpha_1^{-1}\bar{\beta}(\alpha_1, \alpha) = p_1'(0) > p_2'(0) = \alpha_2^{-1}\bar{\beta}(\alpha_2, \alpha)$. By the definition of $\bar{\beta}(\alpha)$

$$\lim_{y\to\infty} p(y, \alpha_1, p_1'(0)) = \lim_{y\to\infty} p(y, \alpha_2, p_2'(0)) = a, \qquad \lim_{y\to\infty} p(y, \alpha_2, p_2'(0)) < \lim_{y\to\infty} p(y, \alpha_2, p_2'(0)).$$

By what has been proved,

$$\lim_{y\to\infty} p(y, \alpha_2, p_1'(0)) \leq \lim_{y\to\infty} p(y, \alpha_1, p_2'(0))$$

and finally,

$$\lim_{y\to\infty} p(y, \alpha_2, p_2'(0)) < \lim_{y\to\infty} p(y, \alpha_1, p_1'(0)),$$

which is impossible. The theorem is proved.

We now prove the existence of a solution of the boundary-value problem (1), (1′). To this end we formulate it somewhat differently. Namely, since under the substitution x by $-x$, y by $-y$, equation (1) is not changed, the boundary-value problem (1), (1′) is equivalent to the following problem.

Find two solutions of equation (1) satisfying the following conditions:

$$y_1(0) = y_2(0) = 0, \qquad y_1'(0) = y_2'(0), \qquad y_2''(0) = cy_2''(0),$$

$$\lim_{x\to+\infty} y_1'(x) = a, \qquad \lim_{x\to+\infty} y_2'(x) = b, \qquad a < b, \quad a \geq 0, \quad b > 0.$$

Denote $\varphi_1(\alpha) = \bar{\beta}(a, \beta)\alpha^{-1}$, $\varphi_2(\alpha) = -c\alpha^{-1}\bar{\beta}(a, \alpha)$ $(a \leq \alpha \leq b)$. As we have shown, $\varphi_1(\alpha)$ decreases and is continuous, $\varphi_2(\alpha)$ does not decrease. If there exists $a < \alpha_0 < b$ such that $\varphi_1(\alpha_0) = \varphi_2(\alpha_0)$, the problem is solved. Suppose this is not so. Then there exist two sequences of numbers (a_n, b_n) such that $a_n < b_n$, $\varphi_2(a_n) < \varphi_1(a_n)$, $\varphi_2(b_n) > \varphi_1(b_n)$, $b_n - a_n < 1/2^n$. If $\alpha_0 = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then $\lim \varphi_2(a_n) = A \leq a_1(\alpha_0)$, $\lim \varphi_2(b_n) = B \geq \varphi_1(\alpha_0)$. By the defini-

By the definition of $\varphi_1(a)$ and $\varphi_2(a)$, we have $\lim_{x\to\infty} y'(x, \alpha_0 - A/c\alpha_0) = \lim_{x\to\infty} y'(x, \alpha_0, -B/c\alpha_0) = a$, and, by theorem (2), $\lim_{x\to\infty} y'(x, \alpha_0, -\varphi_1(\alpha_0)/c\alpha_0) = a$, which, together with $\lim_{x\to\infty} y'(x, \alpha', \varphi_1(\alpha_0)/\alpha_0) = b$, solves the problem.

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Note: Figure translations are in progress. See original paper for figures.