MATHEMATICS

I. M. MELNIK

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Abstract

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I. M. MELNIK

ON THE HILBERT BOUNDARY-VALUE PROBLEM

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§ 1. Let D be a domain (finite or infinite) bounded by a simple smooth contour L, and let $a(t)$, $b(t)$, $c(t)$ be real functions of points of the contour L, satisfying the Hölder condition, with $a(t)$ and $b(t)$ not vanishing anywhere on L. Suppose that the function $\alpha(t)$ maps the contour L homeomorphically and with preservation of the direction of traversal onto itself, has a derivative $\alpha'(t)$ satisfying the Hölder condition and not vanishing anywhere on L; and that the conditions

$$\alpha[\alpha(t)] = t, \qquad a(t)a[\alpha(t)] = b(t)b[\alpha(t)]. \tag{1}$$

are fulfilled.

Determine the function

$$f(z) = u(z) + iv(z),$$

analytic in D and continuously extendable to the contour L, whose real and imaginary parts satisfy the boundary condition

$$a(t)u[\alpha(t)] + b(t)v(t) = c(t) \quad \text{on } L. \tag{2}$$

The finite domain bounded by the contour L will be denoted by D^+, and the infinite one by D^-. Everywhere in the present note we shall assume that a direction of traversal is specified on L, positive with respect to the domain D^+ and, consequently, negative with respect to D^-.

We shall call the formulated problem interior if D is a finite domain, and exterior if D is an infinite domain.

§ 2. **Interior problem.** Assuming that D is a finite domain, and using conditions (1), we transform problem (2) into the equivalent problem

$$f^+[\alpha(t)] = G(t)\overline{f^+(t)} + g(t) \quad \text{on } L, \tag{3}$$

where

$$G(t) = \frac{b(t)}{ia(t)}, \qquad g(t) = \frac{c(t)b[\alpha(t)] + ia(t)c[\alpha(t)]}{a(t)b[\alpha(t)]}. \tag{4}$$

For the case where D is a disk, problem (3) was considered by E. G. Khasabov (unpublished).

Let us first consider the case when the boundary condition (3) has the form

$$f^+[\alpha(t)] = \overline{f^+(t)} + g(t). \tag{5}$$

We shall seek the solution of problem (5) in the form

$$f(z) = \frac{1}{2\pi i} \int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\, d\tau, \tag{6}$$

$\varphi(t)$ is a certain function satisfying the Hölder condition and such that

$$\varphi(t) + \overline{\varphi[\alpha(t)]} = 0. \tag{7}$$

On the basis of the Sokhotski formulas and condition (7), from (6) we obtain the Fredholm integral equation

$$K\varphi \equiv \varphi(t) + \frac{1}{2\pi i} \int_L \left(\frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)} - \frac{1}{\tau - t} + \frac{2i}{\tau'} \frac{\cos(r, n)}{r} \right) \varphi(\tau)\, d\tau = g(t),$$

where $r = |\tau - t|$; n is the inner normal to L at the point τ; (r, n) is the angle between the directions $\overline{\tau t}$ and n; $\tau' = dt/ds$; s is the arc abscissa.

Arguing analogously to how this was done in the work of D. A. Kveselava (^{1}) in solving Carleman' s problem, one can show that the homogeneous equation $K\varphi = 0$ has no nontrivial solutions. Consequently, the nonhomogeneous equation $K\varphi = g(t)$ has a unique solution. Since every integrable solution of the equation $K\varphi = g(t)$ satisfies the Hölder condition and possesses property (7), the corresponding function $f(z)$, defined by formula (6), is a solution of problem (5).

Lemma. If a function $\Phi(z)$, analytic in D^+ and continuously extendable to L, satisfies the boundary condition

$$\Phi^+[\alpha(t)] = \overline{\Phi^+(t)} \quad \text{on } L,$$

then it is a real constant in D^+.

This lemma remains valid also for a function $\Phi(z)$ analytic in D^-, continuously extendable to L, and satisfying the boundary condition

$$\Phi^-[\alpha(t)] = \overline{\Phi^-(t)} \quad \text{on } L.$$

On the basis of the lemma, the general solution of problem (5) is given by the formula

$$f(z) = A + \frac{1}{2\pi i} \int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\, d\tau,$$

where A is an arbitrary real constant; $\varphi(t)$ is the solution of the equation $K\varphi = g(t)$.

Taking into account that $\operatorname{Ind} G(t) = 0$, on the basis of what has been said we obtain that the general solution of problem (3) and of the equivalent problem (2) is given by the relation

$$f(z) = \left\{ A + \frac{1}{2\pi i} \int_L \frac{\psi[\alpha(\tau)]}{\tau - z}\, d\tau \right\} e^{\Gamma(z)}, \tag{8}$$

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\, d\tau, \tag{8*}$$

$\varphi(t)$ is the solution of the equation

$$K\varphi = \ln \left\{ \frac{b(t)}{ia(t)} \right\};$$

$\psi(t)$ is the solution of the equation

$$K\varphi = g(t) e^{-\Gamma^+[\alpha(t)]}.$$

§ 3. **The exterior problem.** Let now D be an infinite domain. Analogously to the preceding case, we first consider the problem

$$f^-[\alpha(t)] = \overline{f^-(t)} + g(t) \quad \text{on } L. \tag{9}$$

We shall seek the solution of the latter problem in the form

$$f(z) = \frac{1}{2\pi i} \int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\, d\tau + ic, \tag{10}$$

where $\varphi(t)$ is a certain function satisfying the Hölder condition and relation (7); c is a real constant.

Using Sokhotskii' s formulas and condition (7), from (9) and (10) we have

$$K^0\varphi \equiv \varphi(t) - \frac{1}{2\pi i}\int_L \left(\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)} - \frac{1}{\tau - t} + \frac{2i}{\tau'}\frac{\cos(r,n)}{r}\right)\varphi(\tau)\,d\tau = 2ic - g(t).$$

It can be shown that the equation $K^0\varphi = 0$ has one linearly independent solution $\varphi(t) = b$, where b is a constant.
The equation adjoint to $K^0\varphi$,

$$K'^0\psi \equiv \psi(t) + \frac{1}{2\pi i}\int_L \left(\frac{\alpha'(t)}{\alpha(\tau)-\alpha(t)} - \frac{(\bar{t}')^2}{\bar{\tau} - \bar{t}}\right)\psi(\tau)\,d\tau = 0$$

also has one linearly independent solution $\psi(t)$, different from the trivial one, and moreover

$$\int_L \psi(t)\,dt \neq 0.$$

Arguing now analogously to the preceding case, we obtain that the general solution of problem (9) is expressed by the relation

$$f(z) = A + ic + \frac{1}{2\pi i}\int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\,d\tau,$$

where $\varphi(t)$ is the solution of the equation $K^0\varphi = 2ic - g(t)$. It can be shown that

$$\operatorname{Re}\left\{\int_L g(t)\psi(t)\,dt \Big/ \int_L \psi(t)\,dt\right\} = 0,$$

therefore the real constant c is determined from the relation

$$2ic\int_L \psi(t)\,dt = \int_L g(t)\psi(t)\,dt.$$

Using the solution of problem (9), we obtain that the general solution of the exterior problem (3), or of the equivalent problem (2), has the form

$$f(z) = \left\{A + ic + \frac{1}{2\pi i}\int_L \frac{r[\alpha(\tau)]}{\tau - z}\,d\tau\right\} e^{\Gamma(z)}, \tag{11}$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\,d\tau + im;$$

$\varphi(t)$ is the solution of the equation

$$K^0 a = -\ln\left\{\frac{b(t)}{ia(t)}\right\} + 2im;$$

$r(t)$ is the solution of the equation

$$K^0\varphi = -g(t)e^{-\Gamma^-[\alpha(t)]} + 2ic;$$

the real constants m and c are determined from the equations

$$2im\int_L \psi(t)\,dt = \int_L \ln\left\{\frac{b(t)}{ia(t)}\right\}\psi(t)\,dt,$$

$$2ic\int_L \psi(t)\,dt = \int_L g(t)\psi(t)e^{-\Gamma^-[\alpha(t)]}\,dt.$$

Both in (8) and in (11), the expression $Ae^{\Gamma(z)}$ is a solution of the corresponding homogeneous problem.

4. Consider the equation

$$a(t)u[\alpha(t)] - \frac{b(t)}{2\pi}\int_0^{2\pi} u(\tau)\operatorname{ctg}\frac{\sigma - s}{2}\,d\sigma = c(t), \tag{12}$$

where $t = e^{is}$, $\tau = e^{i\sigma}$; $a(t)$, $b(t)$, $c(t)$, and $\alpha(t)$ mean the same as in (2).

We shall seek the solution of equation (12) in the class of functions satisfying the Hölder condition. Using the Hilbert formula

$$v(t) = -\frac{1}{2\pi}\int_0^{2\pi} u(\tau)\operatorname{ctg}\frac{\sigma - s}{2}\,d\sigma + \frac{1}{2\pi}\int_0^{2\pi} v(\tau)\,d\sigma,$$

we reduce the solution of equation (12) to the solution of the boundary-value problem (2). Setting

$$\int_0^{2\pi} v(\tau)\,d\sigma = 0, \tag{13}$$

from (12) we obtain the interior boundary-value problem (2) for a function $f(z) = u(z) + iv(z)$ analytic in the unit disk. If the solution $f(z)$ of the homogeneous problem

$$a(t)u[\alpha(t)] + b(t)v(t) = 0 \tag{14}$$

satisfies condition (13), then the homogeneous equation

$$a(t)u[\alpha(t)] - \frac{b(t)}{2\pi}\int_0^{2\pi} u(\tau)\operatorname{ctg}\frac{\sigma - s}{2}\,d\sigma = 0 \tag{15}$$

has the unique solution

$$u(t) = A\ \operatorname{Re} e^{\Gamma^+(t)},$$

depending on one arbitrary real constant A. Here the function $\Gamma^+(t)$ is defined by formula (8*). If the solution of problem (14) does not satisfy condition (13), then the homogeneous equation (15) has no nontrivial solutions.

In the case where condition (13) is not fulfilled, the nonhomogeneous equation (12) has the unique solution

$$u(t) = \operatorname{Re}\left\{\left(A + \frac{1}{2}\psi[\alpha(t)] + \frac{1}{2\pi i}\int_L \frac{\psi[\alpha(\tau)]}{\tau - t}\,d\tau\right) e^{\Gamma^+(t)}\right\}, \tag{16}$$

where $\psi(t)$ means the same as in relation (8); the real constant A is determined from the condition

$$\operatorname{Im}\left\{\int_0^{2\pi}\left(A + \frac{1}{2}\psi[\alpha(t)] + \frac{1}{2\pi i}\int_L \frac{\psi[\alpha(\tau)]}{\tau - t}\,d\tau\right) e^{\Gamma^+(t)}\,ds\right\} = 0.$$

If the solution of problem (14) satisfies condition (13), then a solution of the nonhomogeneous equation (12) exists if and only if the equality

$$\operatorname{Im}\left\{\int_0^{2\pi}\left(\psi[\alpha(t)] + \frac{1}{\pi i}\int_L \frac{\psi[\alpha(\tau)]}{\tau - t}\,d\tau\right) e^{\Gamma^+(t)}\,ds\right\} = 0,$$

holds, and it is given by formula (16).

Rostov-on-Don
State University

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.