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Abstract

Full Text

MATHEMATICS

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On Volterra Operators with an Imaginary Component of One Class or Another

(Presented by Academician A. N. Kolmogorov, 16 III 1961)

1. In the present note we shall adhere to the terminology and notation of the preceding note ⁽¹⁾. In particular, by \mathfrak{S}_p ($1 \leq p < \infty$) we shall denote the Banach space of all operators $A \in \mathfrak{S}_\infty$ (\mathfrak{S}_∞ is the set of all completely continuous operators acting in the Hilbert space \mathfrak{H}) for which

$$(|A|_p)^p = \sum_n s_n^p(A) = \text{Sp}((A^*A)^{p/2}) < \infty.$$

L. A. Sakhnovich ⁽²⁾ (see also ⁽³⁾) showed that, if the imaginary component $A_J (= (A - A^*)/2i)$ of a Volterra operator belongs to the class \mathfrak{S}_2 , then the real component $A_R (= (A + A^*)/2)$ belongs to the same class, and $|A_R|_2 = |A_J|_2$. Subsequently, the authors ^(1, 3, 4) obtained various characteristics of the component A_R of a Volterra operator A with $A_J \in \mathfrak{S}_1$. In the autumn of 1959, at the Fifth All-Union Conference on Functional Analysis in Baku, while discussing with S. G. Krein a number of results reported by the authors, it became clear that the following holds:

Theorem 1. *If one of the components of a Volterra operator A has finite order ≥ 1 , then the other component has the same order.*

Let us explain that the **order** $\rho(A)$ ($0 \leq \rho(A) < \infty$) of an operator A is the lower bound of the numbers p for which $A \in \mathfrak{S}_p$.

For the case $1 < \rho(A_J) < 2$, this theorem is derived from the theorem of L. A. Sakhnovich ⁽²⁾ and Theorem 2 of ⁽³⁾ with the aid of the following lemma.

Lemma 1. *Let T_z be an operator-function with values in \mathfrak{S}_∞ , holomorphic in the strip $a_1 \leq \text{Re } z \leq a_2$. If*

$$|T_{a_j+iy}| \leq C_j \quad (-\infty < y < \infty; j = 1, 2; 1 \leq p_1 < p_2 \leq \infty),$$

then for every intermediate x ($a_1 < x < a_2$) one has

$$|T_{x+iy}|_{p_x} \leq C_1^{t_x} C_2^{1-t_x},$$

where

$$p_x^{-1} = t_x p_1^{-1} + (1 - t_x) p_2^{-1}, \quad t_x = (x - a_1)/(a_2 - a_1).$$

This lemma, which is a peculiar development of the well-known three-lines theorem in the theory of functions, was obtained by the authors jointly with S. G. Krein, to whom the authors are also indebted for assistance in clarifying the role of this lemma in the questions of interest to them.

The validity of Theorem 1 for the case $2 < \rho(A_J) < \infty$ follows already from the preceding on the basis of the general Theorem 2 of ⁽¹⁾.

2. Recently V. I. Matsaev ⁽⁵⁾ developed a new method for investigating the dependence between the behavior of the Hermitian components of a Volterra operator,

based on subtle theorems of function theory already established by him ⁽⁶⁾. In particular, this method enabled him to obtain the following refinement of Theorem 1:

Theorem 2. Let A be a Volterra operator. If $A_J \in \mathfrak{S}_p$, ($1 < p < \infty$), then $A_R \in \mathfrak{S}_p$ and

$$|A_R|_p \leq C_p |A_J|_p, \quad (1)$$

where C_p is a certain constant depending only on p .

Theorem 2 from ⁽¹⁾ also makes it possible to assert that

$$C_p = C_{p/(p-1)} \quad (1 < p < \infty),$$

if by C_p one understands the sharp constant in (1).

After the result of V. I. Matsaev, the authors discovered that only a little needs to be added to their methods in order to obtain Theorem 2, moreover with the following characteristics of the sharp constant C_p :

- 1) the constant C_p is a logarithmically convex (and therefore continuous) function of the argument $\xi = p^{-1}$ on the interval ($0 < \xi < 1$);
- 2) as a function of the argument p , the constant C_p is monotone and increases without bound on the interval ($2 \leq p < \infty$);
- 3) the growth of the constant C_p is bounded by the inequality $C_{2p} \leq C_p + \sqrt{1 + C_p^2}$ ($1 < p < \infty$), and also by the inequalities $\alpha p \leq C_p \leq \beta p$ ($2 \leq p < \infty$), where α and β are certain constants (in particular, one may take $\alpha = 1/2\pi$ and $\beta = (e^{2/3} \ln 2)^{-1} (< 1)$).

Proof of Theorem 2 and of properties 1), 2), 3) of the constant C_p .
Denote by \mathfrak{A} the totality of all self-adjoint finite-dimensional operators. Let \mathfrak{P} be an arbitrary continuous chain (see ⁽¹⁾).

If $H \in \mathfrak{A}$, then the integral

$$\mathfrak{S}(H) = \mathfrak{S}(H; \mathfrak{P}) = i \int_{\mathfrak{P}} (PH dP - dP HP)$$

converges, and $\mathfrak{S}(H) \in \mathfrak{S}_p$ ($p > 1$), while $\mathfrak{S}(H) + iH$ is a Volterra operator with its own chain \mathfrak{P} .

Put

$$\gamma_p = \sup_{H \in \mathfrak{A}} (|\mathfrak{S}(H)|_p / |H|_p) \quad (1 < p < \infty; \gamma_p \leq \infty).$$

Together with the operator $\mathfrak{S}(H) + iH$, the operator $B = (\mathfrak{S}(H) + iH)^2$ is also Volterra and has the chain \mathfrak{P} .

Writing that $B_R = \mathfrak{S}(B_J)$ gives the identity *

$$[\mathfrak{S}(H)]^2 = H^2 + \mathfrak{S}(H\mathfrak{S}(H) + \mathfrak{S}(H)H). \quad (2)$$

Since for any $X, Y \in \mathfrak{S}_{2p}$, $|XY|_p \leq |X|_{2p}|Y|_{2p}$, it follows that

$$|[\mathfrak{S}(H)]^2|_p = (|\mathfrak{S}(H)|_{2p})^2 \leq (|H|_{2p})^2 + 2\gamma_p |H|_{2p} |\mathfrak{S}(H)|_{2p}.$$

Hence we find

$$|\mathfrak{S}(H)|_{2p} / |H|_{2p} \leq \gamma_p + \sqrt{1 + \gamma_p^2},$$

which gives

$$\gamma_{2p} \leq \gamma_p + \sqrt{1 + \gamma_p^2}.$$

From what has been proved and from the quoted theorem of L. A. Sakhnovich it follows that $\gamma_{2^n} < \infty$ ($n = 1, 2, \dots$).

* We point out that the idea of establishing and using the identity (2) arose with the authors in connection with the remarkably simple proof, proposed by M. Kotlyar ⁽⁷⁾, of M. Riesz' s theorem on the Hilbert transform.

Starting from any operator $H \in \mathfrak{K}$, we form, by means of its polar representation $H = UH_1$ (U unitary, and H_1 a nonnegative operator), the holomorphic operator-valued function $T_z = \mathfrak{S}(UH_1^z)$. Application to this operator-valued function of Lemma 1 with $a_1 = p/p_1$, $a = p/p_2$ ($1 < p_1 < p < p_2 < \infty$) shows that, since $\gamma_{p_1} < \infty$ and $\gamma_{p_2} < \infty$, we also have $\gamma_p < \infty$, and, moreover, γ_p is a logarithmically convex function of p^{-1} ($p_1 \leq p \leq p_2$). Since $\gamma_2 = 1$ and $\gamma_{2^n} < \infty$ ($n = 2, 3, \dots$), it follows that $\gamma_p < \infty$ for $2 \leq p < \infty$.

From the indicated properties of the constant γ_p it is already easy to derive that it has properties 1), 2), 3), formulated for the constant C_p .

Let us show that Theorem 2 is valid for $C_p = \sup \gamma_p$, where the supremum is taken over all continuous chains \mathfrak{P} . Let A be an arbitrary Volterra operator. Without loss of generality one may suppose that the proper chain \mathfrak{P} of the operator A is continuous. Obviously,

$$A_R = \mathfrak{S}(A; \mathfrak{P}).$$

Let $\{H_n\}_1^\infty$ be a sequence of operators from \mathfrak{K} such that

$$|H_n - A_J|_p \rightarrow 0.$$

Using (1) and the closedness of the operator $\mathfrak{S} = \mathfrak{S}(\cdot; \mathfrak{P})$, we obtain

$$|\mathfrak{S}(H_n) - A_R| \rightarrow 0.$$

Passing to the limit as $n \rightarrow \infty$ in the relation

$$|\mathfrak{S}(H_n)|_p \leq C_p |H_n|_p$$

we obtain (1) in the general case.

Properties 1), 2), 3) of the constant C_p , obviously, follow from the corresponding properties of the constant γ_p .

- Let $\Pi = \{\pi_n\}_1^\infty$ be an arbitrary nonincreasing sequence of positive numbers π_n such that

$$\lim \pi_n = 0, \quad \sum \pi_n = \infty.$$

By \mathfrak{S}_Π we shall denote the Banach space of all operators $A \in \mathfrak{S}_\infty$ for which

$$|A|_\Pi = \sup \left(\left(\sum_1^n s_k(A) \right) / \sum_1^n \pi_k \right) < \infty,$$

and by $\mathfrak{S}_\Pi^{(0)}$ the subspace of \mathfrak{S}_Π which is the closure in \mathfrak{S}_Π of the set \mathfrak{K} of all finite-dimensional operators. It turns out that the space $\mathfrak{S}_\Pi^{(0)}$ consists of all operators $A \in \mathfrak{S}_\infty$ for which

$$\lim \left(\sum_1^n s_k(A) / \sum_1^n \pi_k \right) = 0.$$

Denote by \mathfrak{S}_π the Banach space of all operators A ($\in \mathfrak{S}_\infty$) for which

$$|A|_\pi = \sum_1^\infty \pi_k s_k(A) < \infty.$$

By Theorem 1, from (1) it follows that the space \mathfrak{S}_π is conjugate to $\mathfrak{S}_\Pi^{(0)}$, while the space \mathfrak{S}_Π is conjugate to \mathfrak{S}_π .

If the sequence $\{\pi_n\}$ is **regular**, i.e.

$$\sum_1^n \pi_k = O(n\pi_n),$$

then the operators $A \in \mathfrak{S}_\Pi$ are characterized by the fact that

$$s_n(A) = O(\pi_n),$$

and the operators $A \in \mathfrak{S}_\Pi^{(0)}$ by the fact that

$$s_n(A) = o(\pi_n) \quad (n \rightarrow \infty).$$

In what follows we shall suppose that

$$\pi_n = n^{-\chi} L(n) \quad (n = 1, 2, \dots; 0 < \chi < 1),$$

where $L(\nu)$ ($1 \leq \nu < \infty$) is a positive continuously differentiable slowly varying function. The last term means that

$$\nu L'(\nu)/L(\nu) \rightarrow 0, \quad \nu \rightarrow \infty.$$

It is easily verified that such a sequence $\{\pi_n\}$ is regular.

V. I. Matsaev ⁽⁵⁾ showed that if, for a Volterra operator A ,

$$s_n(A_J) = O(\pi_n) \quad (1/2 < \chi < 1),$$

then also

$$s_n(A_R) = O(\pi_n),$$

and this assertion remains valid if in it O is replaced by o .

It turns out that the assertion of V. I. Matsaev holds for all χ ($0 < \chi < 1$). Moreover, the following holds:

Theorem 3. Let A be a Volterra operator with $A_J \in \mathfrak{S}$, where \mathfrak{S} is one of the spaces $\mathfrak{S}_\Pi^{(0)}$, \mathfrak{S}_π , \mathfrak{S}_Π . Then $A_R \in \mathfrak{S}$, and

$$|A_R|_{\mathfrak{S}} \leq M |A_J|_{\mathfrak{S}},$$

where M is a constant depending only on χ ($0 < \chi < 1$) and the function $L(\nu)$.

Corollary. If, for a Volterra operator A ,

$$s_n(A_J) = O(\pi_n),$$

then

$$s_n(A_R) = O(\pi_n),$$

and

$$M' \sup(s_n(A_J)/\pi_n) \leq \sup(s_n(A_R)/\pi_n) \leq$$

$$\leq M'' \sup(s_n(A_J)/\pi_n),$$

where M' , M'' are constants depending only on x and the function $L(v)$. If $s_n(A_J) = o(\pi_n)$, then also $s_n(A_R) = o(\pi_n)$.

The proof of Theorem 3 has much in common with the proof of Theorem 2; however, it is considerably more complicated. It uses the theorem of V. I. Matsaev cited above, identity (2), a theorem of the type of Theorem 2 from [1], and a lemma analogous to Lemma 1 for a special system of spaces.

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References

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* In our note "On the theory of triangular representations of non-self-adjoint operators" (1), the following misprints occurred:

	Printed	Should read
p. 1034, line 10	$ A _p^n$	$ A _p$
p. 1034, line 14	$\sup_n \left(\sum_1^n s_k(A) \sum_1^n \pi_k \right)$	$\sup_n \left(\sum_1^n s_k(A) / \sum_1^n \pi_k \right)$
p. 1036, line 7	$(P - Q) \times (P - Q) = 0$	$(P - Q)X(P - Q) = 0$
p. 1037, line 9	$X \in \mathfrak{S}_\infty$	$X \in \mathfrak{S}_\infty$

Note: Figure translations are in progress. See original paper for figures.

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