THE RIEMANN BOUNDARY-VALUE PROBLEM FOR \(n\) PAIRS OF FUNCTIONS WITH MEASURABLE COEFFICIENTS AND ITS APPLICATION TO THE STUDY OF SINGULAR INTEGRALS IN THE SPACES \(L_p\) WITH WEIGHTS

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1961

SovietRxiv

Abstract

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MATHEMATICS

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THE RIEMANN BOUNDARY-VALUE PROBLEM FOR n PAIRS OF FUNCTIONS WITH MEASURABLE COEFFICIENTS AND ITS APPLICATION TO THE STUDY OF SINGULAR INTEGRALS IN THE SPACES L_p WITH WEIGHTS

(Presented by Academician V. I. Smirnov, 5 VI 1961)

Let C be a collection of simple closed Lyapunov contours without common points, bounding a certain domain D^+; by D^- we shall denote the (in general disconnected) domain which completes $D^+ + C$ to the whole plane. The space of vectors (matrices) consisting of n functions on the contour C, summable to the power p, will be denoted by L_p^n $(L_p^{n,n})$. If a vector (matrix) $\Phi^\pm$ of functions analytic in $D^\pm$ and representable by a Cauchy integral has limiting angular values $\Phi^\pm(t)$ belonging to L_p^n $(L_p^{n,n})$ $(p > 1)$, then we shall write $\Phi \in E_p^{n\pm}$ $(E_p^{n,n\pm})$.

We formulate the problem.

Find functions (vectors) $\Phi^\pm(z)$, analytic in $D^\pm$, belonging to the class $E_p^\pm$ $(E_2^{n\pm})$, and satisfying on the contour C the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \tag{1}$$

where $G(t)$ is a given function (matrix); $g(t)$ is a given function (vector) belonging to L_p (L_2^n).

We consider problem (1) separately in two cases: $n = 1$ and $n > 1$.

In the case $n = 1$ the coefficient $G(t)$ satisfies the conditions: 1) $G(t)$ is a measurable function; 2) $0 < M_1 \leq |G(t)| \leq M_2 < \infty$; 3) there exists a $\delta(> 0)$ such that for each point t_0 of the contour C there is a neighborhood in which the values $G(t)$ are contained in a sector with vertex at the origin and opening

$$(2\pi - \delta)/\max\left[p, \frac{p}{p-1}\right] \quad (p > 1).$$

The class of functions $G(t)$ satisfying conditions 1), 2), and 3) will be called the class $A(p)$.

In the case $n > 1$, with respect to the matrix $G(t)$ we make the following assumptions: 1) $G(t)$ has as its elements bounded measurable functions; 2) $G(t)$ can be represented in the form of a product

$$G(t) = G_1(t)G_2(t)G_3(t), \tag{2}$$

where $G_1(t)$ and $G_3(t)$ are continuous in t and have determinants nowhere vanishing; $G_2(t)$ satisfies the condition

$$\operatorname{Re} G_2(t) = \tfrac{1}{2}\left[G(t) + G^*(t)\right] \geq \nu > 0,$$

where G_2^* is the adjoint (transpose with complex-conjugate elements) matrix of G_2; ν is a positive number independent of t; the sign $\geq$ denotes comparison of Hermitian matrices. Matrices satisfying

satisfying the conditions listed above will be called matrices of the class $A^n(2)$, and the quantity $\dfrac{1}{2\pi}\{\arg \Delta(G_1G_3)\}$ will be denoted by $\operatorname{Ind} G$. We note that this definition is justified, since it does not depend on the representation (2).

Let us introduce the following definitions. The space of solutions of the homogeneous problem (1) will be denoted by $A(G)$, and its dimension by $\alpha(G)$; the space of functionals whose orthogonality to the free term g constitutes necessary and sufficient conditions for solvability will be denoted by $B(G)$, and its dimension by $\beta(G)$; the difference $\alpha(G) - \beta(G)$ will be called the index of the problem and denoted by $\chi(G)$.

In terms of the numbers α and β it is convenient to formulate the results of investigations of problem (1). The investigation of problem (1) for one pair of functions ($n = 1$), as usual, differs from the investigation in the case of many functions ($n > 1$) by a greater definiteness of the results. For the problem with one pair of functions, as in work ([8]), one succeeds in determining the increment of the argument $\{\arg G\}_C$ and obtaining the following result.

Theorem 1. The numbers $\alpha(G)$ and $\beta(G)$ are finite and are computed by the formulas:

1) if $\{\arg G\}_C \geqslant 0$,

$$\alpha = \frac{1}{2\pi}\{\arg G\}_C, \qquad \beta = 0, \qquad \chi = \alpha;$$

2) if $\{\arg G\}_C \leqslant 0$,

$$\alpha = 0, \qquad \beta = -\frac{1}{2\pi}\{\arg G\}_C = -\chi.$$

This theorem is a generalization of the results of the works ([1–10]). On the basis of the interpolation theorem of Simuyuki Koizumi ([11]), Theorem 1 remains valid also in the case of Orlicz spaces.

Theorem 1′. Let L_M^* be Orlicz spaces (see ([12])) and let the function M, beginning with certain u_0 $(u > u_0)$, satisfy the condition

$$\min\left[p, \frac{p}{p-1}\right] + \varepsilon_1 \leqslant \frac{uM'(u)}{M(u)} \leqslant \max\left[p, \frac{p}{p-1}\right] - \varepsilon_2 \qquad (\varepsilon_1, \varepsilon_2 > 0),$$

where M' is the right derivative.

Then Theorem 1 remains valid if $g(t) \in L_M^*$ and the solution is sought in the class of functions representable by a Cauchy integral, with angular boundary values from the class L_M^*.

In the course of investigating the problem we obtain results which, in our opinion, are of independent interest. First let us introduce

Definition. The class of functions (matrices) $\rho(t)$ that generate bounded operators

$$A\varphi = \rho(t) \int_C \rho^{-1}(\tau) \frac{\varphi(\tau)}{\tau - t}\, d\tau, \tag{3}$$

acting from $L_p(L_p^n)$ into $L_p(L_p^n)$, will be denoted by W_p (W_p^n) (n is the order of the matrices). If the operator (3) acts from L_M^* into L_M^*, then we shall write $\rho \in W_M^*$. Let us introduce the notion of the local oscillation $\omega(\psi, t)$ of a real function ψ at the point t:

$$\omega(\psi, t) = \inf_l \left\{ \sup_{t \in l} \psi(t) - \inf_{t \in l} \psi(t) \right\},$$

where l ranges over all open arcs containing t.

In the terms introduced by the definitions, the results indicated above are formulated as follows:

Theorem 2. Functions $\rho(t)$ of the form

$$\rho(t) = \left| \exp\left[\frac{1}{2\pi} \int_C \frac{\psi(\tau)}{\tau - t}\, d\tau \right] \right|,$$

where ψ is a real measurable function, $\sup_t \omega(\psi, t) \leqslant 2\pi/\lambda - \delta$, $\lambda \geqslant 2$, $\delta > 0$, belong to the class W_p for p from the segment

$$\left[\frac{\lambda}{\lambda - 1}; \lambda \right].$$

Theorem 2′. Under the assumptions of Theorem 2, $\rho(t)$ belongs to W_M, where M satisfies the condition

$$\frac{\lambda}{\lambda-1} \le \frac{uM'(u)}{M(u)} \le \lambda.$$

Theorem 3. Let $\rho(t)$ satisfy the conditions of Theorem 2, and let M satisfy the conditions of Theorem 2′. Consider the space of functions periodic on the interval $[0,2\pi]$, with norm $\|\varphi\|_{\rho,M} = \|\rho\varphi\|_M$. Then the partial sums $\varphi_n(x)$ of the Fourier series of the function $\varphi(x)$ converge in the norm $\| \; \|_{\rho,M}$ to the original function $\varphi(x)$, i.e. $\|\varphi_n - \varphi\|_{\rho,M} \to 0$.

Theorem 4. If $\psi(t)$ is continuous, then

$$\rho(t) = \left|\exp\left[\frac{1}{2\pi}\int_C \frac{\psi(\tau)}{\tau - t}\,d\tau\right]\right| \in W_p \quad \text{for any } p > 1;$$

$\rho \in W_M$, where M satisfies the conditions

$$1 < \beta \le \frac{uM'(u)}{M(u)} \le \alpha < \infty.$$

This theorem supplements the theorem of V. I. Smirnov ([13]), according to which $\rho \in L_p$ for any $p > 1$.

Numerous works by Soviet and foreign authors ([14], [2], [15], [9]) are connected with the study of singular integrals in spaces with weights. Of these works, those that considered the boundary-value problem proceeded from theorems on weights to the study of the problem. In our work, on the contrary, the Riemann problem is first investigated by methods of functional analysis, and on the basis of the investigation carried out we obtain classes of weights not previously considered.

For the system of Riemann boundary-value problems we have obtained the following results.

Theorem 5. The numbers $\alpha(G)$ and $\beta(G)$ are finite, and their difference is computed by the formula

$$\varkappa(G) = \alpha(G) - \beta(G) = \operatorname{Ind} G.$$

In the special case when the contour C is either a circle or a straight line and the matrix $G(t)$, for every $t \in C$, has a positive definite real part, a more definite result holds.

Theorem 6. Problem (1) is unconditionally and uniquely solvable.

Introduce the following definition: by a canonical factorization of the matrix $G(t)$ we mean its representation in the form

$$G(t) = X^{+}U\,(X^{-})^{-1},$$

where the matrices X^{+}, X^{-}, and U satisfy the following requirements: 1) the elements of the matrices $X^{\pm}$ and $(X^{\pm})^{-1}$ have the form $\varphi^{\pm}(z) + c$, where $\varphi^{\pm} \in E_2^{\pm}$, and c is a constant; 2) the matrix U is diagonal and has the form

$$U = \left\| \begin{matrix} z^{\varkappa_1} & 0 & \\ & \ddots & \\ 0 & & z^{\varkappa_n} \end{matrix} \right\|;$$

3) $X^{\pm}(t)$ and $[X^{\pm}(t)]^{-1}$ belong to W_2^n.

The numbers $\varkappa_1, \varkappa_2, \dots, \varkappa_n$, arranged in decreasing order, will, as usual, be called the partial indices.

Theorem 7. Every matrix of the class $A''(2)$ admits a canonical representation, and the system of partial indices $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ does not depend

of the method of representation and

$$\varkappa_1 + \varkappa_2 + \dots + \varkappa_n = \operatorname{Ind} G; \qquad \sum_{\varkappa_i > 0} \varkappa_i = \alpha(G); \qquad \sum_{\varkappa_i < 0} |\varkappa_i| = \beta(G).$$

Systems of Riemann boundary-value problems were considered in works ([14, 16–19]) under the assumptions of continuity of the elements of the matrix G.

Remark 1. At those points where the matrix $G(t)$ satisfies the Hölder condition, the canonical functions $X^{\pm}(t)$ also satisfy the Hölder condition. This makes it possible to consider the Riemann problem in the case when the matrix G on some arcs of the contour C belongs to the class $A^n(2)$, and on others is piecewise continuous in the sense of Hölder. In this case, at points of discontinuity the classes of solutions may be chosen in the same way as in ([16]).

Remark 2. Theorems 1 and 5 remain valid in the case of the problem with a shift

$$\Phi^{+}[\alpha(t)] = G(t)\Phi^{-}(t) + g(t),$$

where $\alpha(t)$ maps the contour onto itself while preserving the direction of traversal, and $\alpha'(t)$ exists and satisfies the Hölder condition. Moreover, the results of Theorems 1 and 5 can in an obvious way be applied to singular integral equations and to other questions connected with the Riemann boundary-value problem.

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Received
5 VI 1961

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Note: Figure translations are in progress. See original paper for figures.