SPECIAL TYPES OF PAIRS OF \(T\)-CONGRUENCES

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Abstract

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MATHEMATICS

O. S. REDOZUBOVA

SPECIAL TYPES OF PAIRS OF T-CONGRUENCES

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1. We consider pairs of T-congruences ([1]) for which, on the corresponding rays, the focal distances are equal and the angles between the focal planes are equal. The vertex O of a rectangular trihedron is placed on the ray of the congruence of common perpendiculars; the vector $\mathbf{e}_3$ is directed along the ray. The components ω^i, ω_i^j of the infinitesimal displacement of the trihedron $\{O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}$ are determined by the equations

$$d\overrightarrow{O} = \mathbf{e}_i\omega^i, \qquad d\mathbf{e}_i = \omega_i^j\mathbf{e}_j \qquad (i, j = 1, 2, 3),$$

where $\omega_i^j = -\omega_j^i$, $\omega_i^i = 0$, since $\mathbf{e}_i$ are unit vectors.

Such pairs may be of two types. Pairs of T-congruences of the first type are determined by the system of equations

$$\rho_1 = \rho_2, \qquad \rho_1' = \rho_2',$$

$$A_k = H_{k3} + \Omega_{k3}\frac{\rho_1 + \rho_1'}{h_1 - h_2}, \tag{1}$$

$$\frac{\Omega_k^* + h_k\Omega_{k3}^*}{\sin(\alpha_1 - \alpha_2)} = \Omega_{k3}\frac{\rho_1\rho_1'}{h_1 - h_2} \qquad (k = 1, 2).$$

Here h_k denote the abscissas of the points of intersection of the rays of the pair with the rays of the congruence of common perpendiculars, measured from the point O; α_k are the angles that the rays of the pair form with the vector $\mathbf{e}_1$; ρ_k, ρ_k' are the abscissas of the foci F_k, F_k' of the congruences of the pair, measured from the points of intersection with the rays of the congruence of common perpendiculars. Here

$$\Omega_k^* = \omega^1 \sin\alpha_k - \omega^2\cos\alpha_k, \qquad \Omega_k = \omega^1\cos\alpha_k + \omega^2\sin\alpha_k,$$

$$\Omega_{k3}^* = -\omega_1^3\sin\alpha_k + \omega_2^3\cos\alpha_k, \qquad \Omega_{k3} = \omega_1^3\cos\alpha_k + \omega_2^3\sin\alpha_k,$$

$$A_k = \frac{\omega_1^2 + d\alpha_k}{\sin(\alpha_1 - \alpha_2)}, \qquad H_k = \frac{\omega^3 + dh_k}{h_1 - h_2}.$$

Pairs of T-congruences of the first type exist with arbitrariness of 4 functions of one argument. From the equations of the system it follows that *each pair of corresponding rays is equally inclined to the bisectors of the focal planes and is at equal distances from the center of the ray of the congruence of common perpendiculars*, i.e., the pairs are **symmetric**. It is known ([2]) that they are stratifiable. For such pairs: 1) the distance between the boundary points of the congruence of common perpendiculars is equal to the ratio of the distance between the corresponding rays of the pair to the sine of the angle between them; 2) the product of the abscissas of the foci is equal to the product of the square of the distance between the boundary points of the congruence of common perpendiculars and the sines of the angles formed by each of the rays of the pair with the focal planes of this congruence.

If pairs of T-congruences of the 1st type are orthogonal, then they exist with an arbitrariness of 9 arbitrary constants.

Congruences of pairs of the 1st type cannot be normal.

Pairs of T-congruences of the 1st type whose rays intersect the rays of the congruence of common perpendiculars at the centers exist with an arbitrariness of 3 functions of one argument. Such pairs have, respectively, equal distances of the rays of the supplementary congruences F_1F_2 and $F_1'F_2'$, and also equal angles between the focal planes of the corresponding rays of these congruences.

2. Pairs of T-congruences of the 2nd type are determined by the system of equations

$$\rho_1 = -\rho_2', \qquad \rho_2 = -\rho_1', \qquad H_k = A_k,$$

$$\frac{\Omega_k^* + h_k\Omega_{k3}^*}{\sin(\alpha_1 - \alpha_2)} = H_k(\rho_1 - \rho_2) - \rho_1\rho_2 h_1 \frac{\Omega_{k3}}{h_2}. \tag{2}$$

Such pairs of congruences exist with an arbitrariness of 1 function of 2 arguments and possess the property that the focal distances of the rays of the supplementary congruences are equal to one another. The angles between the focal planes of these rays possess the same property.

If pairs of T-congruences of the 2nd type have constant distances between corresponding rays, then they exist with an arbitrariness of 3 functions of one argument. For such pairs one has $H_1 = H_2$, and the following is true:

Theorem 1. *In order that pairs of T-congruences of the 2nd type have constant angles between corresponding rays, it is necessary and sufficient that the distance between these rays be constant.*

Orthogonal pairs of T-congruences of the 2nd type exist with an arbitrariness of 3 functions of one argument.

If both congruences of the pair are normal, then to the system of equations (2) there is adjoined the equation

$$\rho_1\, d\rho_2 + \rho_2\, d\rho_1 = 2(H_1 - H_2)\frac{(h_1 - h_2)^2\{1 - \cos(\alpha_1 - \alpha_2)\}}{\sin^2(\alpha_1 - \alpha_2)}. \tag{3}$$

A pair of such congruences exists with an arbitrariness of 3 functions of one argument.

An interesting special solution is the case $H_1 = H_2$. For such pairs the distance between the corresponding rays of the pair is constant; on the other hand, the angle between them is also constant. Then it follows from equation (3) that $\rho_1\rho_2 = \text{const}$. The converse is true.

Theorem 2. *In order that pairs of T-normal congruences with equal focal distances and angles between the focal planes possess the property that the angles and distances between corresponding rays are constant, it is necessary and sufficient that the product of the abscissas of the foci of each ray be constant.*

Such pairs exist with an arbitrariness of 2 functions of one argument.

If the corresponding rays of the congruences of a pair of the 2nd type intersect at the centers the rays of the congruence of common perpendiculars, then such pairs are pairs of T of the 1st type, and, conversely, if the rays of congruences of a pair T of the 1st type intersect the rays of the congruence of common perpendiculars at the centers, then they are pairs of T of the 2nd type. For pairs of T of the 1st and 2nd types the following theorem is true:

Theorem 3. *In order that pairs of T-congruences with equal focal distances and angles between focal planes be simultaneously pairs of the 1st and 2nd types, it is necessary and sufficient that the rays of the congruence of common perpendiculars intersect the corresponding rays at the centers.*

3. We consider pairs of T-congruences whose corresponding rays intersect the rays of the congruence of common perpendiculars at the centers.

Such pairs are determined by the system of equations

$$A_1 = \frac{\rho_2}{\rho_1}H_1, \qquad A_2 = \frac{\rho_1}{\rho_2}H_2,$$

$$\frac{\Omega_k^* + h_k\Omega_{k3}^*}{\sin(\alpha_1 - \alpha_2)} = -\Omega_{k3}\frac{\rho_1\rho_2}{h_1 - h_2}. \tag{4}$$

They exist with arbitrariness in 4 functions of one argument and have the following properties: 1) the pairs are symmetric and focalized; 2) the focal distances of the rays of the supplementary congruences, as well as the angles between their focal planes, are equal to one another; 3) the distance between the boundary points of the congruence of common perpendiculars is equal to the ratio of the distance between the corresponding rays to the sine of the angle between them; 4) the product of the abscissas of the foci of each of the corresponding rays is equal to the product of the square of the distance between the boundary points of the congruence of common perpendiculars and the sines of the angles formed by each of the rays of the pair with the focal planes of this congruence.

In order that the pairs T, whose corresponding rays intersect at their centers with the rays of the congruence of common perpendiculars, have equal focal distances, it is necessary and sufficient that the angles between their focal planes also be respectively equal to one another.

A special solution leads to pairs of T-congruences simultaneously of the 1st and 2nd types.

State Pedagogical Institute
named after V. I. Lenin

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CITED LITERATURE

[1] S. P. Finikov, *Theory of Pairs of Congruences*, 1956. [2] O. S. Redozubova, *Scientific Notes of the Moscow State Pedagogical Institute named after V. I. Lenin*, **108** (1957).

Note: Figure translations are in progress. See original paper for figures.