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Abstract

Full Text

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ON THE DETERMINATION OF THE DOMAIN OF CAPTURE FOR A SYSTEM CLOSE TO HAMILTONIAN

(Presented by Academician P. S. Aleksandrov on 21 II 1961)

The investigation of certain physical processes ([1,2]) leads to the following problem. Let a system of differential equations with small parameter ε be given:

$$\dot{x} = -\frac{\partial H}{\partial y} + \varepsilon f(x, y, t, \varepsilon), \qquad \dot{y} = \frac{\partial H}{\partial x} + \varepsilon g(x, y, t, \varepsilon), \qquad (I_\varepsilon)$$

where $H = H(x, y)$ is an analytic function of x and y, and $f(x, y, t, \varepsilon)$ and $g(x, y, t, \varepsilon)$ are analytic functions of x, y, ε, continuous together with the first derivative with respect to t, and periodic in t with period 2π. Suppose further that the point (x_c, y_c) is an equilibrium position of center type of the system (I_0) (the system (I_0) is obtained from the system (I_ε) for $\varepsilon = 0$). This means (see ([3]), p. 301) that there exists a neighborhood of the point (x_c, y_c), containing no other equilibrium positions, such that the trajectory issuing from any point of this neighborhood distinct from (x_c, y_c) is closed. We shall be interested in the case when the closed trajectories of the system (I_0) do not fill the whole (x, y)-plane. Moreover, we shall assume that the maximal neighborhood of the point (x_c, y_c) filled with closed trajectories of the system (I_0) lies in a bounded part of the plane. Then, as is known ([4]), there exists a boundary separating on the (x, y)-plane the closed trajectories from the nonclosed ones. This boundary consists of a finite number of trajectories, each of which as $t \to \pm\infty$ tends to some equilibrium position of saddle type. (Here, as everywhere below, it is assumed that all equilibrium positions of the system (I_0) are simple, i.e. if at the point (x_r, y_r)

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0,$$

then

$$\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x\,\partial y}\right)^2 \neq 0).$$

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The problem is, for an arbitrarily chosen instant of time t_0, to find in the (x_0, y_0)-plane the set of points from which, at $t = t_0$, oscillatory solutions issue.

Theorem 1. *Let (x_s, y_s) be an arbitrary equilibrium position of saddle type of the system (I_0). Then there exists a change of variables of the form*

$$x = x_s + a\,(u + \varepsilon p(t, \varepsilon)) \cos\varphi - \frac{1}{a}\,(v + \varepsilon q(t, \varepsilon)) \sin\varphi,$$

$$y = y_s + a\,(u + \varepsilon p(t, \varepsilon)) \sin\varphi + \frac{1}{a}\,(v + \varepsilon q(t, \varepsilon)) \cos\varphi,$$

where a and φ are constants, and the functions $p(t, \varepsilon)$ and $q(t, \varepsilon)$ are analytic in ε in a neighborhood of $\varepsilon = 0$, periodic in t with period 2π, and possess continuous partial derivatives with respect to t up to the second order inclusive, such that in the new variables the system (I_ε) takes the form:

$$\dot{u} = \lambda v - \frac{\partial \widetilde{H}}{\partial v} + \varepsilon \tilde{f}(u, v, t, \varepsilon), \qquad \dot{v} = \lambda u + \frac{\partial \widetilde{H}}{\partial u} + \varepsilon \tilde{g}(u, v, t, \varepsilon), \qquad (II_\varepsilon)$$

where $\lambda > 0$, $f(0, 0, t, \varepsilon) \equiv g(0, 0, t, \varepsilon) \equiv 0$, and the expansion of the function $\bar{H} = \bar{H}(u, v)$ in a neighborhood of the point $(0, 0)$ begins with terms of no lower than third order.

We shall call the system (Π_ε) a **standard form of the system** (I_ε) in a neighborhood of the saddle. The theorem just stated makes it possible to formulate the following basic definition.

A solution $(u_\varepsilon(t), v_\varepsilon(t))$ of the system (Π_ε) will be called a **boundary** solution if it is defined for all t greater than some t_0, $|u_\varepsilon(t)| + |v_\varepsilon(t)| \to 0$ as $t \to \infty$, and there exists a time t_1 such that for all $t > t_1$ the conditions

$$\frac{d}{dt}|u_\varepsilon(t)| < 0 \quad \text{and} \quad \frac{d}{dt}|v_\varepsilon(t)| < 0$$

are satisfied. This definition, obviously, also has meaning for complex values of the parameter ε. Using Theorem 1, it is not difficult to establish what is to be understood by a boundary trajectory of the system (I_ε).

For an arbitrarily prescribed time t_0, denote by $\Gamma_\varepsilon(t_0)$ the set of points in the (x_0, y_0)-plane such that, for $t = t_0$, boundary trajectories of the system (I_ε) issue from them.

Theorem 2. Let $\Delta = \widetilde{E}^2 \setminus \Gamma_\varepsilon(t_0)$, where $\widetilde{E}^2$ is an arbitrary bounded part of the (x_0, y_0)-plane from which the isolated points of the set $\overline{\Gamma}_\varepsilon(t_0) \setminus \Gamma_\varepsilon(t_0)$ have been removed, and let $\Delta' \subset \Delta$ be an arbitrary linearly connected set. Suppose that in the set Δ' there exists a point (x_0, y_0) from which, at $t = t_0$, an oscillatory solution of the system (I_ε) issues. Then, for sufficiently small ε,

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any other solution of the system (I_ε) issuing at $t = t_0$ from an arbitrary point belonging to Δ' will also be oscillatory.

It follows from Theorem 2 that, in order to find the oscillatory solutions of the system (I_ε), it is necessary to find $\Gamma_\varepsilon(t_0)$. With the aid of a certain development of the ideas of paper [5], the following two theorems can be proved.

Theorem 3. There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any complex ε and u_0 satisfying the conditions $|\varepsilon| < \varepsilon_0,\ |u_0| < \delta_0$, and for any t_0, there exists a unique solution $(u_\varepsilon(t), v_\varepsilon(t))$ of the system (Π_ε) such that

$$u_\varepsilon(t_0) = u_0, \quad \frac{d}{dt}|u_\varepsilon(t)| < 0 \quad \text{and} \quad \frac{d}{dt}|v_\varepsilon(t)| < 0$$

for all $t \geq t_0$, and

$$|u_\varepsilon(t)| + |v_\varepsilon(t)| \to 0 \quad \text{as } t \to \infty.$$

Theorem 4. There exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any complex ε and u_0 satisfying the conditions $|\varepsilon| < \varepsilon_1,\ |u_0| < \delta_1$, and for any t_0, a solution $(u_\varepsilon(t), v_\varepsilon(t))$ of the system (Π_ε) satisfying the conditions

$$u_\varepsilon(t_0) = u_0, \quad \frac{d}{dt}|u_\varepsilon(t)| < 0 \quad \text{and} \quad \frac{d}{dt}|v_\varepsilon(t)| < 0$$

for all $t \geq t_0$, and

$$|u_\varepsilon(t)| + |v_\varepsilon(t)| \to 0 \quad \text{as } t \to \infty,$$

has continuous partial derivatives with respect to $\mu = \operatorname{Re}\varepsilon$ and $\nu = \operatorname{Im}\varepsilon$, which satisfy the Cauchy-Riemann conditions for all $t \geq t_0$.

Theorem 4 together with Theorem 1 make it possible to compute the boundary trajectories of the system (I_ε) by expanding them in a series in powers of the parameter ε in a certain, generally speaking, "small" neighborhood of the saddle-type equilibrium position under study. However, using Poincaré' s theorem (see [6], pp. 153-160), one can show that the boundary trajectories of the system (I_ε) will be analytic functions of the parameter ε also in a "large" neighborhood of the saddle-type equilibrium position, since the expansions obtained for the boundary trajectories in powers of the parameter ε will converge also for $t < t_0$ on any finite interval $[t_0 - T, t_0]$ $(T > 0)$, provided ε is sufficiently small. This turns out to be sufficient for proving the following theorem.

Theorem 5. Let G_c be the maximal neighborhood of the point (x_c, y_c) filled with closed trajectories of the system (I_0). Number by the integers from one to n all saddle-type equilibrium positions lying on the boundary of the region G_c, in the order in which they are situated on the boundary when traversing along the boundary of the region G_c in the clockwise direction, and suppose that the motion along the closed trajectories of the system (I_0),

lying in the region G_c, also proceeds clockwise (if this is not so, then this can always be achieved by replacing t by $-t$). Denote by $(x_i(t), y_i(t))$ $(i = 1, 2, \dots, n)$

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the solution of system (I_0) which, as $t \to -\infty$, tends to the i-th equilibrium position of saddle type and, as $t \to +\infty$, tends to the $(i+1)$-st equilibrium position of saddle type (for $i = n$, the $(i+1)$-st equilibrium position is to be understood as the 1st equilibrium position), and let

$$I_i(t_0) = \int_{-\infty}^{\infty} \{f(x_i(t), y_i(t), t - t_0, 0)\dot{y}_i(t) - g(x_i(t), y_i(t), t - t_0, 0)\dot{x}_i(t)\}\, dt$$

be such that

$$\sum_{i=1}^{n} \int_0^{2\pi} I_i(t_0)\, dt_0 > 0.$$

Let, further, for $\delta > 0$, $\overline{G}^-_{c,\delta} \subset G_c$ be the set of points of the region G_c lying at a distance greater than δ from the boundary of the region G_c, and let $G^+_{c,\delta} \supset G_c$ be the δ-neighborhood of the set $\overline{G}_c$. Then, for sufficiently small $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that all solutions of system (I_ε) issuing, at $t = t_0$, from the region $G^-_{c,\delta(\varepsilon)}$ will be oscillatory, and their trajectories for $t \geq t_0$ do not leave the region $G^+_{c,\delta(\varepsilon)}$, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to +0$.

In conclusion, I take this opportunity to express my deep gratitude to S. V. Fomin for his assistance in carrying out this work.

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Note: Figure translations are in progress. See original paper for figures.

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