MECHANICS

1961

SovietRxiv

Abstract

Full Text

MECHANICS

M. Ya. Kushul'

ON THE STABILITY OF FORCED OSCILLATIONS IN AUTOROTATIONAL SYSTEMS

(Presented by Academician I. I. Artobolevskii on 8 VI 1960)

Following (^{1}), we call *autorotational* those potentially self-excited systems whose oscillations occur in two directions (along the axes q and p in the coordinate plane qp). In the present note we shall confine ourselves to the study of nonautonomous autorotational systems of a particular type, described by the equations of motion

$$\ddot{q} + \lambda^2 q = -\alpha\dot{q} - \gamma\delta(\dot{q} + \omega p) - \alpha_1\dot{q}(q^2 + p^2) + \beta\lambda^2 q(q^2 + p^2) + \varepsilon\omega^2\cos\omega t,$$

$$\ddot{p} + \lambda^2 p = -\alpha\dot{p} - \gamma\delta(\dot{p} - \omega q) - \alpha_1\dot{p}(q^2 + p^2) + \beta\lambda^2 p(q^2 + p^2) + \varepsilon\omega^2\sin\omega t. \quad (1)$$

With a certain degree of idealization, such equations describe, in particular, the motion of an unbalanced rotor whose operating speeds lie between the critical speeds of the first and second orders (turbines, spinning spindles, etc.), if the gyroscopic action of the masses may be neglected (^{2}). Under this interpretation, in equations (1) $q(t)$ and $p(t)$ are functions proportional to the projections of the rotor deflection onto fixed coordinate planes; λ is the lowest natural frequency of small oscillations of the rotor; ω is its angular velocity; ε is a quantity determined by the prescribed distribution of unbalance in the rotor system. The nonlinear terms $\alpha_1\dot{q}(q^2+p^2)$ and $\alpha_1\dot{p}(q^2+p^2)$ take into account the increase of the damping decrement with increasing amplitude of oscillation, as indicated by numerous experimental data, while the terms $\beta\lambda^2 q(q^2 + p^2)$ and $\beta\lambda^2 p(q^2 + p^2)$ represent the nonlinear dependence of the curvature of the shaft axis on the displacements. The inelastic stress of the rotor material in equations (1) is taken to be equal to

$$\Delta\sigma = \frac{E\delta}{\Omega^{1-\nu}}\frac{\partial e}{\partial t},$$

where e is the relative strain, Ω is the frequency of the cycles of its variation, and δ is the coefficient of internal friction of the shaft material. According to experimental data, the exponent ν is close to zero over a wide range of frequencies (Bock' s hypothesis). The nonlinear dependence of the inelastic stress on the amplitude of the relative strain is not taken into account here; it can be shown that it will not qualitatively change the results obtained below. The coefficient γ in equations (1) is equal to

$$\gamma = \frac{\lambda^2}{|\omega - \lambda|^{1-\nu}}.$$

Equations (1) admit the particular periodic solution

$$q_0 = d\cos(\omega t + \varphi), \qquad p_0 = d\sin(\omega t + \varphi),$$

$$d = -\frac{\varepsilon\omega^2}{\sqrt{(\omega^2 - \lambda^2 + \beta\lambda^2 d^2)^2 + \omega^2(\alpha + \alpha_1 d^2)^2}}, \qquad \operatorname{tg}\varphi = \frac{\omega(\alpha + \alpha_1 d^2)}{\omega^2 - \lambda^2 + \beta\lambda^2 d^2}, \tag{2}$$

to which purely forced oscillations correspond.

Putting $q = q_0 + x_1$ and $p = p_0 + x_2$, we obtain the following variational equations for solution (2)

$$\begin{aligned} \ddot{x}_1 + \lambda^2 x_1 &= \mu\left(c_{11}x_1 + c_{12}x_2 + c\dot{x}_1\right), \\ \ddot{x}_2 + \lambda^2 x_2 &= \mu\left(c_{21}x_1 + c_{22}x_2 + c\dot{x}_2\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} \mu c_{11} &= 2\beta\lambda^2 d^2 + \alpha_1 d^2\omega\sin 2\omega t + \beta\lambda^2 d^2\cos 2\omega t, \\ \mu c_{12} &= -\gamma\delta\omega + \alpha_1\omega d^2 - \alpha_1\omega d^2\cos 2\omega t + \beta\lambda^2 d^2\sin 2\omega t, \\ \mu c &= -\alpha - \gamma\delta - \alpha_1 d^2, \\ \mu c_{21} &= \gamma\delta\omega - \alpha_1\omega d^2 - \alpha_1\omega d^2\cos 2\omega t + \beta\lambda^2 d^2\sin 2\omega t, \\ \mu c_{22} &= 2\beta\lambda^2 d^2 - \alpha_1\omega d^2\sin 2\omega t - \beta\lambda^2 d^2\cos 2\omega t. \end{aligned}$$

The stability of the zero solution of system (3) depends on the sign of the real part of the characteristic exponents. Applying the substitution $x_s = y_s\exp(\zeta t)$, where $\zeta = \lambda\sqrt{-1} + \mu b + \ldots$ is the desired characteristic exponent, we obtain the system of equations

$$\ddot{y}_1 + 2\lambda\dot{y}_1\sqrt{-1} = \mu y_1\left(c_{11} + c\lambda\sqrt{-1} - 2b\lambda\sqrt{-1}\right) + \mu c_{12}y_2 +$$

$$+ \mu\dot{y}_1(c - 2b) + \mu^2(\ldots) + \ldots,$$

$$\ddot{y}_2 + 2\lambda\dot{y}_2\sqrt{-1} = \mu y_1 c_{21} + \mu y_2\left(c_{22} + c\lambda\sqrt{-1} - 2b\lambda\sqrt{-1}\right) +$$

$$+ \mu\dot{y}_2(c - 2b) + \mu^2(\dots) + \dots,$$

which must admit a periodic solution of period $2\pi/\omega$. From this condition we obtain

$$\left[2\beta\lambda^2 d^2 - \lambda\left(\alpha + \gamma\delta + \alpha_1 d^2 + 2b\right)\sqrt{-1}\right]^2 + \omega^2\left(\gamma\delta - \alpha_1 d^2\right)^2 = 0,$$

whence

$$2b_{1,2} = -\alpha - \gamma\delta - \alpha_1 d^2 \pm \frac{\omega}{\lambda}\left|\gamma\delta - \alpha_1 d^2\right| + \frac{2\lambda^2\beta d^2}{\omega^2}\sqrt{-1}.$$

The other two values of the coefficient b are complex conjugates of b_1 and b_2. Thus, the sole condition for stability of solution (2) will be

$$\mathrm{Re}(2b_2) = -\alpha - \gamma\delta - \alpha_1 d^2 + \frac{\omega}{\lambda}\left|\gamma\delta - \alpha_1 d^2\right| < 0. \tag{4}$$

Let us denote by y_0^2 the quantity

$$\alpha_1 y_0^2 = \gamma\delta\left(\frac{\omega}{\lambda} - 1\right) - \alpha - \alpha_1 d^2\left(\frac{\omega}{\lambda} + 1\right).$$

In the range of variation of the parameter ω where $y_0^2 > 0$, the real part of the characteristic exponent $\mathrm{Re}(2b_2) > 0$, and solution (2) is unstable. The function y_0^2 can be transformed into the form

$$y_0^2 = f_1 - f_2,$$

where

$$f_1 = \frac{\lambda^{1+\nu}\delta}{\alpha_1}\left(\frac{\omega}{\lambda} - 1\right)^{\nu}, \qquad f_2 = \frac{\alpha}{\alpha_1} + \varepsilon^2\frac{(\omega/\lambda)^4}{\left(\omega^2/\lambda^2 - 1\right)^2}\left(\frac{\omega}{\lambda} + 1\right). \tag{5}$$

Illustrative graphs of the functions f_1 and f_2 are given in Fig. 1. The forced oscillations (2) are unstable in the interval of variation of the parameter $\omega_1 < \omega < \omega_2$; to the right of the bifurcation value of the angular velocity ω_2, the forced-oscillation regime again becomes stable. In the interval under consideration, $\omega_1 < \omega < \omega_2$, the instability is caused by internal-friction forces. The possibility of loss of stability of rotors due to internal-friction forces was first pointed out

Fig. 1

Figure 1: Fig. 1

by Kimball (^{3}). The stability conditions for the rectilinear equilibrium form of balanced rotors were then studied in the works of many authors, for example (^{4,5}); in these works it was proved that

the rectilinear form becomes unstable for any angular velocity ω greater than some value ω.

The phenomenon of stabilization of the regime of forced oscillations of rotors to the right of the bifurcation value of the angular velocity ω_2, due to imbalance, has been observed by a number of investigators (see, for example, (^{6})).

However, as is clear from condition (4), with a further increase of the angular velocity ω, a second interval of instability of the solution (2) arises. Indeed, since the last term in expression (4) contains the factor ω/λ, as ω increases there will come a moment when the real part $\mathrm{Re}(2b_2)$ again becomes positive; instability will occur if

$$\frac{\omega}{\lambda} x_1 d^2 > \frac{\omega}{\lambda}\gamma\delta - (\alpha + \alpha_1 d^2 + \gamma\delta).$$

Fig. 1

In many particular cases the lower boundary of the second instability interval is so high that it is practically unattainable.

It is noteworthy that, in contrast to the first interval, the instability in the second interval is caused not by the forces of internal friction but by the nonlinear terms of the external-friction forces, while the forces of internal friction in this case exert a stabilizing effect.

The result obtained may seem somewhat unexpected. We therefore consider it expedient to give one more proof of its validity.

For large values of the parameter ω, the ratio λ/ω is a small quantity. Putting

$$\mu = \frac{\lambda}{\omega}, \qquad \tau = \omega t, \qquad a = \frac{\alpha}{\lambda}, \qquad a_1 = \frac{\alpha_1}{\lambda},$$

we transform system (1) to the form

$$\begin{aligned} \frac{d^2q}{d\tau^2} &= -\mu^2 q - \mu a\frac{dq}{d\tau} - \mu^2(1+\mu+\mu^2+\ldots)\delta\left(\frac{dq}{d\tau}+p\right) \\ &\quad - \mu(q^2+p^2)\left(a_1\frac{dq}{d\tau} - \mu\beta q\right) + \varepsilon\cos\tau, \\ \frac{d^2p}{d\tau^2} &= -\mu^2 p - \mu a\frac{dp}{d\tau} - \mu^2(1+\mu+\mu^2+\ldots)\delta\left(\frac{dp}{d\tau}-q\right) \\ &\quad - \mu(q^2+p^2)\left(a_1\frac{dp}{d\tau} - \mu\beta p\right) + \varepsilon\sin\tau. \end{aligned} \tag{6}$$

The periodic solution of the equations obtained, to within the square of the small parameter, will be

$$\begin{aligned} q_0 &= -\varepsilon(1+\mu^2 k_1)\cos\tau + \mu\varepsilon k_2\sin\tau, \\ p_0 &= -\varepsilon(1+\mu^2 k_1)\sin\tau - \mu\varepsilon k_2\cos\tau, \end{aligned} \tag{7}$$

where

$$k_1 = 1-(a+a_1\varepsilon^2)^2-\beta\varepsilon^2, \qquad k_2 = a+a_1\varepsilon^2.$$

Restricting ourselves to terms containing μ in the first degree, we obtain the variational equations for the solution (7) in the form

$$\frac{d^2x_1}{d\tau^2} = \mu c_{11}x_1 + \mu c_{12}x_2 + \mu c\frac{dx_1}{d\tau}, \qquad \frac{d^2x_2}{d\tau^2} = \mu c_{21}x_1 + \mu c_{22}x_2 + \mu c\frac{dx_2}{d\tau}, \tag{8}$$

where

$$c_{11} = a_1\varepsilon^2\sin 2\tau, \qquad c_{12} = a_1\varepsilon^2(1-\cos 2\tau), \qquad c = -(a+a_1\varepsilon^2), \tag{9}$$

$$c_{21} = -a_1\varepsilon^2(1+\cos 2\tau), \qquad c_{22} = -a_1\varepsilon^2\sin 2\tau.$$

The fundamental equation of the quasiharmonic system (8) for $\mu = 0$ has a zero root of multiplicity four, to which two groups of solutions correspond; therefore, generally speaking, the characteristic roots ζ of system (8) will expand in powers of the quantity $\mu^{1/2}$ (7). Putting $x_s = y_s\exp(\zeta t)$ $(s = 1,2)$, where

$$\zeta = \mu^{1/2}d_1 + \mu d_2 + \cdots,$$

we obtain the equations

$$\frac{d^2y_1}{d\tau^2} = (\mu c_{11} + \mu\zeta c - \zeta^2)y_1 + \mu c_{12}y_2 + (\mu c - 2\zeta)\frac{dy_1}{dt},$$

$$\frac{d^2y_2}{d\tau^2} = \mu c_{21}y_1 + (\mu c_{22} + \mu\zeta c - \zeta^2)y_2 + (\mu c - 2\zeta)\frac{dy_2}{dt};$$

we seek their periodic solution of period 2π in the form of series

$$y_1 = y_1^0 + \mu^{1/2}y_1^{(1)} + \mu y_1^{(2)} + \cdots, \qquad y_2 = y_2^0 + \mu^{1/2}y_2^{(1)} + \mu y_2^{(2)} + \cdots.$$

Equating the coefficients of like powers $\mu^0, \mu^{1/2}, \mu$, we find, from the periodicity condition for the functions y_s,

$$y_1^0 = A_1, \qquad y_2^0 = A_2, \qquad y_2^{(1)} = B_1, \qquad y_2^{(2)} = B_2,$$

and also

$$A_1\int_0^{2\pi}(c_{11} - d_1^2)^2\,dt + A_2\int_0^{2\pi}c_{12}\,dt = 0,$$

$$A_1\int_0^{2\pi}c_{21}\,dt + A_2\int_0^{2\pi}(c_{22} - d_1^2)\,dt = 0,$$

where c_{ij} are determined by formulas (9). From the last equation we find

$$d_1^4 + a_1^2\varepsilon^4 = 0,$$

whence it follows that the motion under investigation is unstable for large values of the parameter ω.

Thus it has been established that the monoharmonic regime of forced oscillations (2) of the autorotational nonautonomous system (1) has two intervals of instability with respect to the parameter ω. In the first interval, instability is caused by the forces of internal friction; in the second, for large values of the parameter ω, by the nonlinear terms of the forces of external friction; the forces of internal friction, on the contrary, exert a stabilizing effect.

Received
31 V 1960

REFERENCES CITED

1. S. P. Strelkov, *ZhTF*, vol. 17 (1939).
2. M. Ya. Kushul, *Izv. AN SSSR, Mekhanika*, No. 1 (1960).
3. A. Kimball, *Gen. Electr. Rew.*, **24** (1924).
4. E. L. Nikolai, *Tr. Leningr. industr. inst.*, No. 6 (1937).
5. F. M. Dimentberg, *Flexural Vibrations of Rotating Shafts*, Moscow, 1959.
6. M. I. Chaevskii, *Izv. AN SSSR, OTN*, No. 11 (1955).
7. M. Ya. Kushul, *Prikl. matem. i mekh.*, No. 10 (1958).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.