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Abstract

Full Text

MATHEMATICS

D. M. GROBMAN

TOPOLOGICAL AND ASYMPTOTIC EQUIVALENCE OF SYSTEMS OF DIFFERENTIAL EQUATIONS

(Presented by Academician P. S. Aleksandrov on 20 V 1961)

In the work ([1]) the topological equivalence of the systems

$$\frac{dx}{dt} = Ax + f(x) \tag{1}$$

$$\frac{dy}{dt} = Ay, \tag{2}$$

was established, where A is a constant square matrix of order n, having no eigenvalues with zero real part; x, y, and $f(x)$ are n-dimensional vectors, and $f(x)$ satisfies, in a neighborhood of the singular point $x = 0$, a Lipschitz condition with a sufficiently small constant. More precisely: it was proved that there exists a homeomorphism $y = \Phi(x)$, mapping a certain neighborhood G_1 of the point $x = 0$ onto a certain domain G_2 and transforming solutions of one system into solutions of the other. Moreover, the corresponding solutions turn out simultaneously to be either O^+-curves, or O^--curves, or saddle curves.

The finer properties of the solutions of system (2), under the assumptions made concerning $f(x)$, are, generally speaking, not inherited by system (1), and the trajectories corresponding by virtue of Φ may have different asymptotics.

However, if near the origin the vector $f(x)$ is subject to requirements more stringent than the Lipschitz condition, namely, if for all x' and x'' small in norm the inequality $*$

$$|f(x') - f(x'')| \le g(r)\,|x' - x''|, \tag{3}$$

holds, where $r = \max\{|x'|,\ |x''|\}$, and as $r \to 0$ $g(r) \to 0$ sufficiently rapidly, then, as is known ([2–6]), the O-curves of systems (1) and (2) are asymptotically equivalent. But even in this case the homeomorphism Φ does not guarantee similarity in the asymptotic behavior of the corresponding trajectories.

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In the present work conditions are stated under which there exists a homeomorphism Φ^* free of this drawback. Before giving the precise formulation of the theorem, let us recall the definitions.

The **characteristic exponent**, or simply the **exponent**, of a vector $x(t)$ is

$$\overline{\lim}_{t\to+\infty}\frac{1}{t}\ln|x(t)|.$$

The **minus-exponent** of a vector $x(t)$ is

$$\overline{\lim}_{t\to-\infty}\frac{1}{t}\ln|x(t)|.$$

* Here $|x| = (x,x)^{1/2}$ is the norm of the vector x. The scalar product is understood to be equal to the sum of the products of the corresponding coordinates.

It is clear that if the exponent (or minus-exponent) is negative, then $x(t)$ tends to 0 as $t \to +\infty$ (or as $t \to -\infty$).

Two vectors $x(t)$ and $y(t)$ are called **analogous** as $t \to +\infty$ (as $t \to -\infty$) if the ratio of their norms tends, as $t \to +\infty$ (as $t \to -\infty$), to unity, while the difference between their direction cosines tends to zero.

The **deviation** of x from y, or simply the **deviation**, is the ratio

$$\frac{|x(t)-y(t)|}{|y(t)|}.$$

It is quite clear that the analogy of $x(t)$ and $y(t)$ is equivalent to the fact that their deviation tends to zero as $t \to \infty$, or to the fact that

$$x(t) = y(t) + z(t), \tag{4}$$

where $|z(t)| = o(|y(t)|)$ as $t \to +\infty$ (or as $t \to -\infty$).

Two systems of differential equations are called **homeomorphic in the domains** G_1 and G_2 if there exists a topological correspondence between G_1 and G_2 under which the trajectories of the first system lying in G_1 go over into trajectories of the second system, and conversely.

Theorem. *If:*

a) *the matrix A has no eigenvalues with zero real parts;*

b) $f(0) = 0$;

c) *in some neighborhood of the point $x = 0$, $f(x)$ satisfies condition (3), and for* $0 < r \le r_0 < 1$

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$$g(r) \leq L_0 \frac{r^\alpha}{|\ln r|^{(2+\alpha)m+1+\beta+\eta}}, \tag{5}$$

where $m+1$ is the number equal to the order of the maximal "box" in the Jordan form of A, and $L_0 \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, $\eta > 0$ are certain constants, then:

1) *systems (1) and (2) are homeomorphic in certain domains containing the origin;*

2) *the corresponding O-curves are analogous;*

3) *the deviation of the corresponding O^+-curves as $t \to +\infty$ is $O\big(e^{\alpha\omega t} t^{-(m+\beta+\eta)}\big)$, where ω is their exponent; for the corresponding O^--curves the deviation as $t \to -\infty$ is $O\big(e^{\alpha\omega|t|}|t|^{-(m+\beta+\eta)}\big)$, where ω is their minus-exponent.*

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