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Abstract

Full Text

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SINGULAR OPERATORS OF ARBITRARY ORDER ON A MANIFOLD

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MATHEMATICS

1. We consider a new class of operators, including differential and singular integral operators on manifolds. This gives a unified approach to the index theory of such operators, making it possible to use the achievements of two different theories.

2. Notation. V is a compact infinitely differentiable manifold of dimension $n > 0$; $x = (x_1, \dots, x_n)$ are local coordinates; $\Xi(V)$ is the bundle of nonzero tangent vectors ξ of the manifold V; $\alpha = (\alpha_1, \dots, \alpha_n)$ is a set of natural numbers, $|\alpha| = \alpha_1 + \dots + \alpha_n$,

$$D^\alpha = i^{-|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}};$$

$|\xi|$ is the absolute value of the tangent vector ξ, defined by some Riemannian metric μ on V.

We consider the following functional spaces: $E(V)$ is the Schwartz space of infinitely differentiable functions on V ([1]); $W_2^{(l)}(V)$ (l an arbitrary integer) is the Sobolev–Leray space of generalized functions on V, the definition of which is transferred in the standard way ([1]) from the Euclidean case ([2]); $E'(V)$ is the Schwartz space of all generalized functions on V ([1]).

Definition. An operator B in the space $E(V)$ will be called a **singular operator of order** $m \geq 0$ if, in any system of local coordinates x, we have

$$Bu(x) = \sum_{|\alpha| \leq m} H_\alpha D^\alpha u(x), \qquad u \in E(V),$$

where H_α are singular integral operators with infinitely differentiable coefficients in the domain of definition of x, metrized by means of μ ([3]).

The closures of the operator B in the spaces $W_2^{(l)}(V)$ and $E'(V)$ will also be denoted by B. Examples of singular operators of order m are differential operators of order m; singular operators of order 0 coincide with singular integral operators.

Let $p \in V$ be the origin in the coordinate system x. The symbol of a singular operator B of order m at the point p is the function, defined on the collection of nonzero tangent vectors ξ at the point p, by

$$\sigma_B(\xi_x) = \sum_{|\alpha|=m} \sigma_{H_\alpha}(\xi_x)\, \xi_x^\alpha,$$

where ξ_x is the representation of the vector ξ as an element of the space dual to the Euclidean coordinate space R_x^n; $\sigma_{H_\alpha}(\xi_x)$ is the symbol of the sin-

singular integral operator H_α (3); $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ ($\xi_1, \dots, \xi_n$ are the components of the vector ξ_x). The **symbol of a singular operator B of order m** will mean the function, so defined, on the whole bundle $\Xi(V)$. The independence of this definition from the choice of local coordinates and of the metric μ follows from the following proposition:

Lemma*. *Let H be a singular integral operator with infinitely differentiable coefficients in the sense of (3), defined on a bounded domain G of Euclidean space R_x^n. Then under a nonsingular infinitely differentiable change of coordinates $x' = x'(x)$ in G, the operator H is transformed into an operator H' of the same type, and*

$$\sigma_{H'}(\xi_{x'}) = \sigma_H\left(\left(\frac{\partial x}{\partial x'}\right)^* \xi_{x'}\right),$$

where $(\partial x/\partial x')^*$ is the matrix adjoint to the Jacobian matrix of the first derivatives of the coordinates x with respect to the coordinates x'.

In the proof of the lemma, the work (4) is used essentially.

Theorem 1 follows immediately from (3).

Theorem 1. 1) *The mapping $B \to \sigma_B(\xi)$ is a homomorphism of the linear space of singular operators of order m onto the linear space of infinitely differentiable positively homogeneous functions of degree m.* 2) *The kernel of this homomorphism consists of operators completely continuous from $W_2^{(m)}(V)$ to $W_2^{(0)}(V)$.* 3) *If the orders of the operators B_1 and B_2 coincide, then $\sigma_{B_1+B_2}(\xi) = \sigma_{B_1}(\xi) + \sigma_{B_2}(\xi)$.* 4) *The product of singular operators B_1 and B_2 is a singular operator and $\sigma_{B_1 B_2}(\xi) = \sigma_{B_1}(\xi)\sigma_{B_2}(\xi)$.*

3. A singular operator of order m will be called **elliptic** if its symbol vanishes nowhere.

Theorem 2. *Let $m \geqslant 1$. In order that a singular operator B of order m be elliptic, it is necessary and sufficient that the a priori estimate*

$$\|u\|_{l+m} \leqslant C(\|Bu\|_l + \|u\|_l), \qquad u \in E(V);$$

hold; here $\| \ \|_k$ denotes the norm in $W_2^{(k)}(V)$; l is an arbitrary integer; C is a certain constant independent of u.

The proof of Theorem 2 is analogous to the proof of a priori estimates for elliptic differential operators (5).

Theorem 3. *For the ellipticity of a singular operator B of order m it is necessary and sufficient that the following set of conditions hold: a) the generalized solutions of the equation $Bu = 0$ are infinitely differentiable; b) these solutions form a finite-dimensional subspace; c) the operator B is normally solvable in $E(V)$, $E'(V)$, and $W_2^{(l)}(V)$ (l is any integer); d) the dimensions of the spaces $E(V)/BE(V)$, $E'(V)/BE'(V)$, $W_2^{(l)}(V)/BW_2^{(l)}(V)$ are finite and equal.*

For $m \geqslant 1$, the necessity of a), ···, d) follows from the existence of a regularizing operator, which is constructed analogously to (6), and from Theorem 2. The case $m = 0$ is reduced to the preceding one by considering operators with symbol $|\xi|\sigma_B(\xi)$.

The sufficiency of the set of conditions a), b), c), d) is verified with the aid of Theorem 2.

Introduce the notation: ν_B is the dimension of the space of solutions of the equation $Bu = 0$; ρ_B is the dimension of the space $E(V)/BE(V)$; $\varkappa_B = \nu_B - \rho_B$ is the index of the operator B.

From Theorem 1 and (7), with the aid of (3), it follows

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* After this paper had been submitted for publication, it became known to the author that an analogous proposition had earlier been established by S. G. Mikhlin (unpublished).

Theorem 4. 1) The index χ_B of an elliptic singular operator of order m is determined by the symbol $\sigma_B(\xi)$. 2) The index of the operator B is stable under uniformly small changes of the first $2n$ derivatives of the symbol B.

We give several theorems on the vanishing of the index.

Theorem 5. If the symbol of an elliptic singular operator B of order m is a real-valued function, then $\chi_B = 0$.

For the proof it suffices to consider the Hilbert space $W_2^{(0)}(V)$. It is easy to see that the operator B^* adjoint to the operator B is singular and $\sigma_{B^*}(\xi) = \overline{\sigma_B(\xi)}$. In our case $\sigma_{B^*}(\xi) = \sigma_B(\xi)$, so that (by Theorem 4) $\chi_B = \chi_{B^*}$. But always $\chi_B = -\chi_{B^*}$.

Theorem 6. If V is an orientable manifold and $n \geqslant 3$, then the index of an arbitrary elliptic singular operator B of order m is equal to zero.

Let $\Omega(p)$ be the set of unit tangent vectors at the point $p \in V$. By virtue of the simple connectedness of $\Omega(p)$, one can select a single-valued continuous branch of the function $\log \sigma_B(\xi)$ on $\Omega(p)$. Let

$$b(p) = \exp \int_{\Omega(p)} \log \sigma_B(\xi)\, d\Omega(p),$$

where $d\Omega(p)$ is the normalized volume element of $\Omega(p)$. The function $b(p)$ is single-valued and nowhere vanishes on V. It is proved that on Ξ there exists a single-valued branch, continuous in ξ and t, of the function $\{\sigma_B(\xi/|\xi|)b^{-1}(p)\}^t$, $0 \leqslant t \leqslant 1$. Hence, and from Theorem 4, it follows that the index of the operator B is equal to the index of the operator with symbol $b(p)|\xi|^m$. From Theorem 5 it follows that it is equal to zero.

4. Consider a $p \times p$ matrix $\mathfrak{B}$ of singular operators of order m. The matrix $\mathfrak{B}$ defines a singular operator $\mathfrak{B}$ of order m in the space $(E'(V))^p$. The symbol $\sigma_{\mathfrak{B}}(\xi)$ of this operator is called the matrix of symbols of the elements of the matrix $\mathfrak{B}$. We shall call the operator $\mathfrak{B}$ **elliptic** if the determinant of the matrix $\sigma_{\mathfrak{B}}(\xi)$ nowhere vanishes.

For elliptic singular operators $\mathfrak{B}$ of order m in the space $(E'(V))^p$, the theorems corresponding to Theorems 1, 2, 3, and 4 are valid.

Theorem 5 carries over to elliptic operators with self-adjoint symbol. Hence follows the

Remark. The index of an elliptic singular operator $\mathfrak{B}$ of order m coincides with the index of the singular integral operator with symbol $|\xi|^{-m}\sigma_{\mathfrak{B}}(\xi)$.

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Note: Figure translations are in progress. See original paper for figures.