ON THE FORM OF EXTREMAL FUNCTIONS OF ONE CLASS, LEAST DEVIATING FROM ZERO AND SATISFYING SEVERAL RELATIONS LINEAR WITH RESPECT TO THE COEFFICIENTS

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Abstract

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MATHEMATICS

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ON THE FORM OF EXTREMAL FUNCTIONS OF ONE CLASS, LEAST DEVIATING FROM ZERO AND SATISFYING SEVERAL RELATIONS LINEAR WITH RESPECT TO THE COEFFICIENTS

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1. The authors of numerous works on the theory of functions of a given form, monotone on a certain interval and least deviating from zero on another, generally speaking, interval have until now restricted themselves mainly to the case of intervals symmetric with respect to the origin, considering chiefly either the interval $[-1,+1]$ or the entire real axis. However, if one abandons the requirement of symmetry of the intervals, one can pose and solve a number of new extremal problems, in particular for functions monotone on the real half-axis.

2. Let two intervals $[a,b]$ and $[c,d]$ of the real axis be given (the intervals may also be infinite), with the interval $[c,d]$ containing the interval $[a,b]$. Denote by

   $$y_n(x)=\sum_{k=0}^{n} p_k x^k$$

   a nonnegative polynomial on $[c,d]$, of degree not exceeding n, with real coefficients, and consider the functions

   $$f_n(x)=\int_a^x (x-z)^h p(z) y_n(z)\,dz, \tag{1}$$

   where $h \geq 0$ is a given integer; $p(z)$ is a given summable and nonnegative function on $[c,d]$ (in the case of an infinite interval $[c,d]$ we assume that all products $z^k p(z)$, $k=0,1,\dots,n+h$, are summable).

If the points a and c coincide, the functions (1) will be multiply monotone of order $h+1$ on the interval $[c,d]$. If $a>c$, we shall assume that h is an even integer. In this case all derivatives of the functions (1) of odd orders up to the $(h+1)$-st order inclusive will be nonnegative on $[c,d]$.

As special cases of the class of functions (1) (for $p(z)\equiv 1$) we obtain the known classes $T_N^{(h)}$ and $B_N^{(h/2)}$ of polynomials (3).

Let the coefficients of the polynomials $y_n(x)$ satisfy s linearly admissible relations of the form

$$\omega_j\{y_n\} \equiv \sum_{k=0}^{n} p_k a_{kj} = A_j, \qquad j = 1, 2, \dots, s, \tag{2}$$

where a_{kj} and A_j are given real numbers, and

$$\sum_{j=1}^{s} |A_j| \neq 0.$$

The problem is posed as follows: among all polynomials $y_n(x)$, nonnegative for $x \in [c, d]$ and satisfying the relations (2), choose one such that the osc-

scillation of the function (1) on the interval $[a, b]$

$$\mathcal{L}^{(h+1)}\{y_n\} = \int_a^b (b-x)^h p(x) y_n(x)\, dx \tag{3}$$

be minimal.

To solve this problem it is important to establish the form of the polynomials among which the extremal polynomial is contained (i.e., the polynomial realizing the minimum of the integral (3)). Our communication is devoted to clarifying this question.

3. Since the polynomials $y_n(x)$ are nonnegative for $x \in [c, d]$, they can always be represented in the form

$$y_n(x) = \varphi(x) u_m^2(x) q_r(x), \tag{4}$$

where:

1) $\varphi(x)$ is a polynomial of degree not exceeding two, nonnegative on $[c, d]$, whose roots can only be the values $x = c$ and $x = d$ (and the roots will be simple);
2) $u_m(x)$ is a polynomial of degree not exceeding m, all of whose roots are real and belong to the interval $[c, d]$;
3) $q_r(x)$ is a polynomial of degree $\leq r$, all of whose real roots are located outside the interval $[c, d]$.

4. If $s = 1$ and $f_n(x) \in T_N^{(h)}$, or $s = 2$ and $f_n(x) \in B_N^{(h/2)}$, then, as B. A. Rymarenko proved $(^{1-3})$, for extremal polynomials $r = 0$, i.e. $q_r(x) \equiv 1$. The theorems of B. A. Rymarenko extend, as is easy to see, also to the general case of functions (1), if $s = 1$ and the interval $[c, d]$ is finite, and $s = 2$, when this interval coincides with the entire real axis.* For any larger

number of relations (2), for extremal polynomials, generally speaking, one will have $r > 0$.

By directly carrying over S. N. Bernstein' s method of proof (^{4}) of the theorems on the form of extremal nonnegative trigonometric polynomials in the presence of one or two relations between the coefficients, it is not difficult to prove that in our problem, for extremal polynomials,

$$0 \leq r < s.$$

5. Let us establish additional relations (besides the given relations (2)) that will be satisfied by the coefficients of the extremal polynomials $y_n^*(x)$. The number of the required relations must be equal to $m + r + 1 - s$ ($y_n^*(x)$ is represented in the form (4)).

With respect to the conditions (2) we additionally assume that the linear forms

$$\sum_{k=0}^{n} p_k \alpha_{kj}, \quad j = 1, 2, \dots, s,$$

and

$$y_n(\eta_i) = \sum_{k=0}^{n} p_k \eta_i^k, \quad i = 1, 2, \dots, t \leq m,$$

of the coefficients p_k are linearly independent. Here $\eta_1, \eta_2, \dots, \eta_t$ are all the real pairwise distinct roots of the equation $y_n(x) = 0$ located inside the interval (c, d).

Under this assumption, from among the relations (2) we exclude relations of the form

$$y_n(\eta) = y_n'(\eta) = \dots = y_n^{(2\nu-1)}(\eta) = 0 \tag{5}$$

(η is some point of the interval (c, d)), but this does not diminish the generality of the problem, since, if among the relations (2) there are 2ν relations (5), then the function (1) can be represented in the form

$$f_n(x) = \int_a^x (x - z)^h (z - \eta)^{2\nu} p(z) y_{n-2\nu}(z)\, dz,$$

where the polynomial $y_{n-2\nu}(x)$, nonnegative on $[c, d]$ and of degree not exceeding $n - 2\nu$, satisfies $s - 2\nu$ relations of the form (2). Thus, from among the relations (2), the relations (5) are excluded.

* B. A. Rymarenko in communication (^{3}) considered the case $p(x) = e^{-x^2}$ and $s = 2$.

We note that, under the stated assumptions concerning conditions (2), there always exists a polynomial (denote it by $\psi(x)$), whose degree does not exceed $2m + r$, satisfying the conditions

$$\omega_j\{\varphi(x)\psi(x)\} = 0, \qquad j = 1, 2, \dots, s;$$

$$\psi(\eta_i) = B_i^2 > 0, \qquad i = 1, 2, \dots, t$$

(the numbers B_i^2 are prescribed arbitrarily).

6\. Let $\tau = m + r - s \geq 0$. Introduce the notation

$$F_l(x; \lambda_1) = u_m(x) \sum_{k=0}^{s} \alpha_{kl} x^{k+l} - \lambda_1 \psi(x), \qquad l = 0, 1, \dots, \tau,$$

where the coefficients α_{kl}, for each value of l, are determined by the system of equations

$$\sum_{k=0}^{s} \alpha_{kl}\, \omega_j\{\varphi(x) u_m(x) x^{k+l}\} = 0, \qquad j = 1, 2, \dots, s. \tag{6}$$

Then the polynomials

$$\tilde{y}_n(x) = y_n^*(x) - \lambda_2 \varphi(x) F_l(x; \lambda_1)$$

also satisfy relations (2). For the time being we subject the parameters λ_1 and λ_2 only to the single condition $\lambda_1 \lambda_2 > 0$.

If, for at least one value of l, the inequality

$$\sum_{k=0}^{s} \alpha_{kl} \int_a^b (b - x)^h p(x) \varphi(x) u_m(x) x^{k+l}\, dx \neq 0 \tag{7}$$

holds, then, by an appropriate choice of the parameter λ_1 (and consequently also of the sign of λ_2), one can always arrange that

$$\mathcal{L}^{(h+1)}\{\tilde{y}_n\} = \mathcal{L}^{(h+1)}\{y_n^*\} - \lambda_2 \int_a^b (b - x)^h p(x) \varphi(x) F_l(x; \lambda_1)\, dx < \mathcal{L}^{(h+1)}\{y^*\}. \tag{8}$$

Fix the value of λ_1 in such a way that inequality (8) is satisfied. It remains now to show that, for sufficiently small $|\lambda_2|$, the inequality

$$\tilde{y}_n(x) \geq 0$$

holds for all $x \in [c, d]$. This is not difficult to verify, taking into account that for $x = \eta_i,\ \ i = 1, 2, \ldots, t$,

$$\tilde{y}_n(\eta_i) = \lambda_1 \lambda_2 \varphi(\eta_i) B_i^2 > 0.$$

Consequently, the polynomial $y_n^*(x)$ will be extremal only in the case when, for all $l = 0, 1, \ldots, \tau$, the equalities

$$\sum_{k=0}^{s} \alpha_{kl} \int_a^b (b-x)^h p(x) \varphi(x) u_m(x) x^{k+l}\, dx = 0 \tag{9}$$

are satisfied.

Eliminating the quantities α_{kl} from the system of equations (6) and (9), we obtain the desired relations between the coefficients of the extremal polynomial:

$$\left| \begin{matrix} \displaystyle\int_a^b (b-x)^h p(x)\varphi(x) u_m(x) x^l\, dx & \cdots & \displaystyle\int_a^b (b-x)^h p(x)\varphi(x) u_m(x) x^{l+s}\, dx \\ \omega_1\{\varphi(x) u_m(x) x^l\} & \cdots & \omega_1\{\varphi(x) u_m(x) x^{l+s}\} \\ \cdots & \cdots & \cdots \\ \omega_s\{\varphi(x) u_m(x) x^l\} & \cdots & \omega_s\{\varphi(x) u_m(x) x^{l+s}\} \end{matrix} \right| = 0, \tag{10}$$

$$l = 0, 1, \ldots, \tau.$$

7. In conclusion we give the solution of two extremal problems for functions of the class (1), based on the application of the system (10).

Problem 1. Find the minimal oscillation on the segment $[-1, +1]$ of the polynomial $P_N(x) \in B_N$ $(N = 2\mu + 1)$ ^{3}, subject to the conditions

$$P_N'(+1) = A^2 > 0, \qquad P_N''(+1) = B, \qquad P_N'''(+1) = C \tag{11}$$

$(A^2, B,$ and C are given numbers).

Problem 2. Let the coefficients of the polynomial $y_n(x)$, nonnegative on the semiaxis $[0, +\infty)$, of degree $\leq n$, satisfy the conditions

$$y_n(0) = A^2 > 0, \qquad y_n'(0) = B \tag{12}$$

(A^2 and B are given numbers). Find the minimal oscillation on $[0, +\infty)$ of the function

$$f_n(x) = \int_0^x z^\nu e^{-z} y_n(z)\, dz,$$

where $\nu \geq 0$ is a given integer.

The minimal oscillation of the polynomial $P_N(x)$ in Problem 1 will be equal to

$$\mathcal{L}_{P_N} = \frac{2 \cdot 3^2}{(\mu+1)^2} \left\{ \left[1 - \frac{(3\mu^2 + 6\mu + 1)^2}{2^2 \cdot 5\mu^2(\mu+2)^2} \right] a^2 + \frac{2^4}{3\mu^2(\mu+2)^2} \left[b^2 - \frac{3(\mu^2 + 2\mu - 1)}{2^2} ab + ac \right] \right\}$$

under the condition

$$\frac{3\mu^2 + 6\mu + 1}{2^3 \cdot 5} a^2 - ab + \frac{2^2}{(\mu-1)(\mu+3)} ac > 0$$

and

$$\mathcal{L}_{P_N} = \frac{2 \cdot 3^2}{(\mu+1)^2} \left\{ a^2 + \frac{2^8}{3\mu^2(\mu+2)^2} b^2 + \frac{2^8 \cdot 5c^2}{(\mu-1)^2\mu^2(\mu+2)^2(\mu+3)^2} - \frac{2^4 ab}{\mu(\mu+2)} + \frac{2^5 \cdot 5ac}{3(\mu-1)\mu(\mu+2)(\mu+3)} - \right.$$

if

$$\frac{3\mu^2 + 6\mu + 1}{2^2 \cdot 5} a^2 - ab + \frac{2^2}{(\mu-1)(\mu+3)} ac \leq 0.$$

Here

$$a = \pm A; \qquad b = \frac{B}{\pm 2A}; \qquad c = \frac{2A^2 c - B^2}{\pm 4A^3}.$$

Analogously, in Problem 2 we obtain for the minimal oscillation (assuming for definiteness that n is an even number: $n = 2\mu$)

$$\mathcal{L}_{f_n} = \frac{(\alpha-1)!\, \alpha!\, (\mu-1)}{(\alpha+\mu-1)!} \left(B + \frac{2\mu\alpha - \alpha + 1}{\alpha^2 - 1} A^2 \right),$$

if

$$\frac{B}{2}+\frac{\mu-1}{\alpha+1}A^2>0,$$

and

$$\mathcal{L}_{f_n}=\frac{(\alpha-1)!(\alpha+1)!(\mu-1)!}{(\alpha+\mu)!}\left\{\frac{\alpha}{4}\frac{B^2}{A^2}+\frac{\mu(\mu\alpha+1)}{\alpha^2-1}A^2+\mu B\right\},$$

if

$$\frac{B}{2}+\frac{\mu-1}{\alpha+1}A^2\leq 0.$$

Here $\alpha=\nu+2$.

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CITED LITERATURE

[1] B. A. Rymarenko, Doctoral dissertation, Institute of Mathematics, Academy of Sciences of the Ukrainian SSR, Kiev, 1951. [2] B. A. Rymarenko, DAN, **103**, No. 3 (1955). [3] B. A. Rymarenko, DAN, **119**, No. 1 (1958). [4] S. N. Bernstein, Izv. AN SSSR, Division of Physico-Mathematical Sciences, No. 5 (1930).

Note: Figure translations are in progress. See original paper for figures.