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Abstract

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ON THE DIFFERENTIAL PROPERTIES OF THE GREEN' S FUNCTION OF A MULTI-POINT BOUNDARY-VALUE PROBLEM

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1. Let the equation be given

$$L[x] \equiv x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = f(t), \tag{1}$$

where $p_1(t), \dots, p_n(t), f(t)$ are assumed to be continuous on some interval $[a, b]$.

We shall consider, for equation (1), boundary conditions of the form

$$l_i[x] = \sum_{k=0}^{n-1} c_{ik} x^{(k)}(a_i) = A_i, \quad i = 1, 2, \dots, n, \tag{2}$$

$$a \leq a_1 \leq a_2 \leq \dots \leq a_n \leq b.$$

The coincidence of several a_i thus means that several functionals are specified at one point; these, naturally, we shall regard as linearly independent.

As is known, the existence and uniqueness of the solution of problem (1)–(2) are equivalent to the absence of nontrivial solutions of the corresponding homogeneous problem

$$L[x] = 0; \tag{3}$$

$$l_i[x] = 0, \quad i = 1, 2, \dots, n, \tag{4}$$

or, what is the same, to the condition

$$\begin{vmatrix} l_1[y_1] & l_1[y_2] & \cdots & l_1[y_n] \\ l_2[y_1] & l_2[y_2] & \cdots & l_2[y_n] \\ \cdot & \cdot & \cdots & \cdot \\ l_n[y_1] & l_n[y_2] & \cdots & l_n[y_n] \end{vmatrix} \neq 0, \tag{5}$$

where $y_1(t), y_2(t), \ldots, y_n(t)$ is any fundamental system of solutions of equation (3).

In what follows we assume that condition (5) is satisfied. For some problems of the type under consideration, solvability conditions were indicated by Vallée-Poussin ([1]) (see also ([8])) and by the author ([2]).

Let us introduce some definitions and notation. The square $a \le t,\ s \le b$ will be denoted by K; the same square with the straight lines of the form $s = a_i$ removed from it will be denoted by K_0; K_0 with the diagonal $t = s$ removed from it—by K_1. The points a_i lying inside the interval (a, b) will be called **interior points**.

By the **order of the functional** l_i we shall mean the order of the highest derivative entering the functional with nonzero coefficient. The order of the point a_i is defined as the greatest of the orders of the functionals specified at a_i.

Let $x_1(t), x_2(t), \ldots, x_n(t)$ be a fundamental system of solutions of equation (3), satisfying the conditions

$$l_i[x_j] = \delta_{ij}, \qquad i,\ j = 1,\ 2,\ \ldots,\ n.$$

The existence of such a system is obviously ensured by condition (5).

By $D_i(t)$ $(i = 1, 2, \ldots, n)$ we shall denote the algebraic cofactor of the element $x_i^{(n-1)}(t)$ in the Wronskian determinant

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ x_1'(t) & x_2'(t) & \ldots & x_n'(t) \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \ldots & x_n^{(n-1)}(t) \end{vmatrix}.$$

For convenience of notation, put also

$$a = a_0, \quad b = a_{n+1}; \qquad x_0(t) \equiv x_{n+1}(t) \equiv D_0(t) \equiv D_{n+1}(t) \equiv 0.$$

2. By the Green' s function of problem (3)–(4) we shall, as usual, mean a function of two variables $G(t, s)$, defined in K and satisfying the following conditions:

1) $G(t, s),\ G_t'(t, s), \ldots,\ G_{t^{n-2}}^{(n-2)}(t, s)$ are jointly continuous in the variables in K_0;

2) $G_{t^{n-1}}^{(n-1)}(t, s)$ is jointly continuous in the variables in K_1, and on the diagonal $t = s$ has a jump equal to one:

$$G_{t^{n-1}}^{(n-1)}(t+0, t) - G_{t^{n-1}}^{(n-1)}(t-0, t) = 1 \qquad (t \ne a_i,\ i = 1,\ 2, \ldots, n);$$

3) $G(t, s)$, as a function of t, satisfies in K_1 conditions (3)–(4).

In the usual way it is established that the solution of problem (1)–(2) is given by the formula

$$x(t) = \int_a^b G(t, s) f(s)\, ds + \sum_{i=1}^{n} A_i x_i(t).$$

Thus, under the boundary conditions (2), the operator L possesses a completely continuous inverse.

Checking the fulfillment of conditions 1)–3) shows that the Green' s function of problem (3)–(4) is determined in K_0 by the formula

$$G(t, s) = \begin{cases} -\dfrac{1}{W(s)} \displaystyle\sum_{i=k+1}^{n+1} D_i(s) x_i(t), & \text{if } t \le s, \\ \dfrac{1}{W(s)} \displaystyle\sum_{i=0}^{k} D_i(s) x_i(t), & \text{if } t > s, \end{cases} \tag{6}$$

$$a_k < s < a_{k+1}, \qquad k = 0,\ 1, \dots,\ n.$$

In order that $G(t, s)$ be defined on the whole square K, it must still be extended in some way to the straight lines $s = a_i$, $i = 0, 1, \dots, n + 1$. It is convenient to extend $G(t, a)$ by continuity with respect to s from the right, and $G(t, b)$ by continuity with respect to s from the left. As for straight lines of the form $s = a_i$, where a_i is some interior point, on these lines $G(t, s)$, generally speaking, has a discontinuity with respect to s, i.e., the identity $G(t, a_i - 0) \equiv G(t, a_i + 0)$ does not in general hold. The manner of extending $G(t, s)$ to these straight lines may be chosen arbitrarily; for definiteness, let it be right-continuity with respect to s.

3. An essential question is that of the differential properties of the Green' s function, i.e., the question of the conditions for the validity of the relation

$$G(t, a_i - 0) = G(t, a_i + 0) \tag{7}$$

and of analogous relations for the derivatives of $G(t, s)$.

The question of certain properties of the Green' s function for broad classes of multipoint boundary-value problems was studied in a number of papers by A. S. Smogorzhevskii ([3,4]), and also in ([5,6]). So far as we know, the assertions on differential properties given below have not been noted.

4. In proving Theorems 1 and 2 formulated below, the following properties of the functions $D_i(t)$ are used.

Lemma 1. The equality $c_{i,n-1} = 0$ is equivalent to the equality

$$D_i(a_i) = 0. \tag{8}$$

The equality $c_{i,n-2} = 0$ is equivalent to the equality

$$D_i'(a_i) = 0. \tag{9}$$

5. From the first assertion of the lemma it follows that $G(t, s)$ is continuous in s on the line $s = a_i$, where a_i is an arbitrary interior point, if and only if the order of a_i is less than $n - 1$. At the same time, together with $G(t, s)$, its derivatives with respect to t up to order $(n - 2)$, inclusive, are continuous in s on the same line.

Thus the following theorem holds.

Theorem 1. In order that $G(t, s)$ be continuous in K with respect to the variables jointly, it is necessary and sufficient that among $a_1, a_2, \dots, a_n$ there be no interior points of order $n - 1$. In this case $G_t'(t, s)$, $G_t''(t, s), \dots, G_t^{(n-2)}(t, s)$ are also continuous in K with respect to the variables jointly.

6. Similarly, for the case $n \geqslant 3$, with the aid of the second assertion of the lemma it is established that the continuity of $G(t, s)$, together with $G_s'(t, s)$, with respect to s on a line of the form $s = a_i$, where a_i is an interior point, holds if and only if the order of a_i is less than $n - 2$.

Thus the following theorem is valid:

Theorem 2. Let $n \geqslant 3$. In order that $G(t, s)$ and $G_s'(t, s)$ be continuous in K with respect to the variables jointly, it is necessary and sufficient that among $a_1, a_2, \dots, a_n$ there be no interior points of orders $n - 1$ and $n - 2$.

7. Let us give an example of an application of the results obtained. A. O. Gel' fond showed ([7]) that the eigenvalues λ_n of the integral equation

$$x(t) = \lambda \int_a^b G(t, s) x(s)\, ds,$$

numbered in order of increasing moduli, satisfy the relation

$$\sum_1^\infty \frac{1}{|\lambda_n|^{\frac{2}{2m+1}+\varepsilon}} < \infty,$$

where ε is any positive number, if the kernel $G(t, s)$ has m derivatives with respect to s, uniformly bounded for $a \leqslant t, s \leqslant b$. Comparing this result of A. O. Gel' fond with Theorem 2 and observing that the eigenvalues of the boundary-value problem

$$L[x] = \lambda x; \tag{10}$$

$$l_i[x] = 0, \qquad i = 1, 2, \dots, n, \tag{11}$$

do not depend on whether it is considered on the interval $[a, b]$ or on the interval $[a_1, a_n]$, we obtain the following proposition:

Theorem 3. Let $n \geqslant 3$, and suppose that inside the interval (a_1, a_n) there are no points of orders $n-1$ and $n-2$. Then, for any $\varepsilon > 0$, the eigenvalues of problem (10)—(11) satisfy the relation

$$\sum_1^\infty \frac{1}{|\lambda_n|^{1/3+\varepsilon}} < \infty.$$

Analogues of Theorems 1—3 can apparently be obtained for problems with boundary conditions of a more general form.

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Note: Figure translations are in progress. See original paper for figures.