MATHEMATICS

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1961

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Abstract

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ON THE STRUCTURE OF SUBSPACES OF A HILBERT SPACE

(Presented by Academician A. I. Mal'tsev, 7 II 1961)

It is known ([6]) that the completion by intersections R of a distributive structure L need not be a Dedekind structure. On the other hand, the completion by intersections of a Boolean algebra is a Boolean algebra. The question remained open whether the completion of a Dedekind structure with complements would necessarily be Dedekind (([2]), Problem 57). In the present note it is shown that even for a Dedekind structure with orthocomplements the completion by intersections need not be Dedekind.

Let H be a Hilbert space. Denote the set of all subspaces of H by $L(H)$, the set of all closed subspaces by $T(H)$, and the set of all finite-dimensional subspaces and their orthogonal complements by $S(H)$. Since a finite-dimensional subspace is closed (([1]), p. 60) and the orthogonal complement of any set is a closed subspace (([1]), p. 83), we have $L(H) \supseteq T(H) \supseteq S(H)$. We shall assume that the sets $L(H)$, $T(H)$, and $S(H)$ are ordered by inclusion.

Proposition 1. *Let H_1 and A be subspaces in H, with H_1 closed and A finite-dimensional. Then $H_1 + A$ is a closed subspace* (([3]), p. 50).

Proposition 2. *Every closed subspace is the orthogonal complement of its orthogonal complement* (([4]), p. 102).

We shall agree to denote by H_1^0 the orthogonal complement of the subspace H_1.

Proposition 3. *If H_1 is a closed subspace, then every element $x \in H$ is uniquely representable in the form $x = x_1 + x_2$, where $x_1 \in H_1$, $x_2 \in H_1^0$* (([4]), p. 101).

If it is necessary to emphasize that the subspace H_1 is finite-dimensional (is the orthogonal complement of a finite-dimensional subspace), then we shall place the symbol $\vee$ ($\wedge$) over the letter H_1.

Let H_1 and H_2 be subspaces, with H_1 closed. On the basis of Proposition 3, every element $x_\alpha \in H_2$ can be represented in the form $x_\alpha = x'_\alpha + x''_\alpha$, where $x'_\alpha \in H_1$, $x''_\alpha \in H_1^0$. The subspace consisting of the elements $x''_\alpha \in H_1^0$ for all $x_\alpha \in H_2$ will be denoted by $H_2^{H_1}$.

Lemma 1. *Let the closed subspace H_1 contain the subspace $\widehat{H}_2$. Then H_1 is the orthogonal complement of a finite-dimensional subspace.*

Proof. Since $H_1 \supseteq H_2$, it follows that $H_1^0 \subseteq H_2^0$. But H_2^0 is finite-dimensional by assumption; hence H_1^0 is also finite-dimensional. It remains to use Proposition 2.

Lemma 2. *The subspace $\widehat{H}_1 + H_2$ is closed, whatever the subspaces $\widehat{H}_1$ and H_2 may be.*

Proof. Let us verify the validity of the relation

$$\hat{H}_1 + H_2 = \hat{H}_1 + H_2^{H_1}.$$

Let $x \in H_1 + H_2$, i.e., $x = h_1 + h_2$ $(h_1 \in H_1,\ h_2 \in H_2)$. Take a representation of the element $h_2 = h_2' + h_2''$ $(h_2' \in H_1,\ h_2'' \in H_1^0)$. Then

$$x = h_1 + h_2 = (h_1 + h_2') + h_2''.$$

But $(h_1 + h_2') \in H_1,\ h_2'' \in H_2^{H_1}$, whence $x \in H_1 + H_2^{H_1}$. Conversely, let $y \in H_1 + H_2^{H_1}$, i.e., $y = h_1 + h_2''$ $(h_1 \in H_1,\ h_2'' \in H_2^{H_1})$. Then for some $h_2 \in H_2$ we have $h_2 = h_2' + h_2''$, where $h_2' \in H_1$, whence

$$y = h_1 + h_2'' = h_1 + h_2 - h_2' = (h_1 - h_2') + h_2 \in H_1 + H_2.$$

Since $H_2^{H_1} \leq H_1^0$ and H_1^0 is finite-dimensional by assumption, $H_2^{H_1}$ is also finite-dimensional. It remains to use Proposition 1.

If in a partially ordered (p.o.) set there exist greatest or least elements, we shall denote them by 1 and 0, respectively.

Lemma 3. Let the p.o. set A lie in the lattice B, which preserves the order relation in A. Then, if

$$x = \sup_B\{x_\alpha\} \quad \left(x = \inf_B\{x_\alpha\}\right)^*,$$

$x \in A$ and $\{x_\alpha\} \subseteq A$, then

$$x = \sup_A\{x_\alpha\} \quad \left(x = \inf_A\{x_\alpha\}\right).$$

Proof is trivial.

Corollary 1. Let the lattice with respect to the order of inclusion U consist of subspaces (closed subspaces) of the space H. Since the set-theoretic intersection of subspaces (closed subspaces) is a subspace (closed subspace), the lattice product of subspaces in U will coincide with the set-theoretic intersection, if this intersection belongs to U.

Definition 1. A lattice L with 1 and 0 will be called a **lattice with orthocomplementations** if there exists a mapping

$$\varphi:\ x \mapsto x'$$

of the lattice L into L, such that: 1) from $x \geq y$ it follows that $x' \leq y'$; 2) $x + x' = 1$; 3) $x \cdot x' = 0$; 4) $x'' = x$ for any $x, y \in L$.

Lemma 4. Let L be a lattice with orthocomplementations. Then from

$$x = \sup\{x_\alpha\} \quad (x = \inf\{x_\alpha\})$$

it follows that

$$x' = \inf\{x'_\alpha\} \quad (x' = \sup\{x'_\alpha\}),$$

where x' and x'_α are the orthocomplements of the elements x and x_α.

Proof. Let $x = \sup\{x_\alpha\}$. Since $x \geq x_\alpha$, we have $x' \leq x'_\alpha$ for all α. Hence x' is a lower bound for $\{x'_\alpha\}$. If y' is another lower bound, i.e., $y' \leq x'_\alpha$, then $y \geq x''_\alpha = x_\alpha$ for all α, whence

$$y \geq x = \sup\{x_\alpha\}.$$

But then $y' \leq x'$, i.e., x' is the exact lower bound for $\{x'_\alpha\}$.

Lemma 5. The p.o. set $T(H)$ is a complete lattice with orthocomplementations.

Proof. Note that $T(H) \ni H$. Since $T(H)$ consists of all closed subspaces of H, by Corollary 1, together with any set of subspaces $\{H_\alpha\}$ in $T(H)$, the element

$$\inf_T\{H_\alpha\}$$

also belongs to $T(H)$. Using (2), p. 82, we conclude that $T(H)$ is a complete lattice.

To each subspace $X \in T(H)$ we put in correspondence its orthogonal complement X^0. The fulfillment of 2) and 4) of Definition 1 follows in an obvious way from Propositions 3 and 2, respectively. The fulfillment of 1) and 3) is easily checked.

Theorem 1. The p.o. set $S(H)$ is a sublattice of the lattice $L(H)$**.

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* $\sup_B$ ($\inf_B$) means that the exact upper (lower) bound is taken in the p.o. set B.

** The p.o. set $L(H)$ is a complete Dedekind lattice ((2), p. 188).

Proof. Let H_1 and H_2 belong to $S(H)$. Denote

$$H_3 = \overset{L}{\inf}(H_1, H_2) = H_1 \cap H_2$$

(the set-theoretic intersection is meant); $H_4 = H_1 + H_2$.

Consider the following cases.

1. H_1, H_2; then H_3 and H_4 are finite-dimensional; hence $H_3, H_4 \in S(H)$.
2. $\check{H}_1, \check{H}_2$; then $H_3 \leq H_1$, and hence H_3 is finite-dimensional and $H_3 \in S(H)$. From Proposition 1 it follows that H_4 is closed, and since $H_4 \geq H_2$, by Lemma 1 we have $H_4 \in S(H)$.
3. $\hat{H}_1, \hat{H}_2$; let
$$H_5 = H_1^0 + H_2^0 = H_1^0 + H_2^0.$$
Since $H_5 \geq H_1^0$ and $H_5 \geq H_2^0$, it follows that
$$H_5^0 \leq (H_1^0)^0 = H_1, \qquad H_5^0 \leq (H_2^0)^0 = H_2,$$
whence
$$H_5^0 \leq H_1 \cap H_2 = H_3.$$
The subspace H_5 is finite-dimensional as the sum of the finite-dimensional subspaces H_1^0 and H_2^0; moreover, H_3 is closed as the intersection of closed subspaces; consequently from $(\hat{H}_5) \leq H_3$ and Lemma 1 it follows that $H_3 \in S(H)$. By Lemma 2 the subspace H_4 is closed, and since, moreover, $H_4 \geq \hat{H}_1$, in view of Lemma 1, the subspace $H_4 \in S(H)$.

We have shown that together with every pair of subspaces H_1 and H_2 from $S(H)$, their structural sum and product from $L(H)$ also belong to $S(H)$, i.e. $S(H)$ is a substructure in $L(H)$.

Corollary 2. The structure $S(H)$ is Dedekind.

Corollary 3. The structure $S(H)$ is a substructure in $T(H)$.

Proof. Let $H_1, H_2 \in S(H)$; then
$$H_3 = \overset{L}{\inf}(H_1, H_2)$$
and
$$H_4 = H_1 + H_2$$
also belong to $S(H)$, since $S(H)$ is a substructure in $L(H)$. From $T(H) \supseteq S(H)$ we conclude that, together with $H_1, H_2 \in S(H)$, the structure $T(H)$ contains also $H_3, H_4 \in S(H)$; whence, in view of $T(H) \subseteq L(H)$ and Lemma 3, we obtain
$$H_3 = \overset{T}{\inf}(H_1, H_2)$$
and $H_4 = H_1 + H_2$, i.e. $S(H)$ is a substructure in $T(H)$.

Corollary 4. The structure $S(H)$ is a structure with orthocomplements.

Proof. The structure $T(H)$, by Lemma 5, possesses orthocomplements, and in the substructure $S(H)$ of the structure $T(H)$, together with every subspace H_1, its orthocomplement in $T(H)$ (the orthogonal complement of H_1) also belongs. It is easy to verify the fulfillment of 1)–4) of Definition 1.

Lemma 6. *Let a partially ordered set P be contained in a complete structure L, and let L preserve the order relation in P. If every element $x \in L$ is the exact lower and exact upper bound for subsets of elements of P, then L is isomorphic to the completion R of the partially ordered set P by cuts.**

Proof. To each $x \in L$ put in correspondence the set

$$I(x) = \{p \in P;\ p \leq x\}:\ x \xrightarrow{\varphi} I(x).$$

Since

$$x = \overset{L}{\sup} I(x),$$

we have

$$I^*(x) = \{p \in P,\ p \geq x\},$$

and then

$$x = \overset{L}{\inf} I^*(x),$$

whence

$$I(x) = I^{*+}(x).$$

Consequently, $I(x) \in R$ for every $x \in L$. Conversely, let $I \in R$ and

$$x = \overset{L}{\sup} I.$$

Then

$$I^* = I^*(x),$$

whence

$$I = I^{*+} = I^{*+}(x) = I(x).$$

Thus φ maps L onto the whole structure R. From $x \geq y$ it follows that $I(x) \geq I(y)$, and conversely. Consequently, the mappings φ and φ^{-1} are one-to-one and isotone, i.e. φ is an isomorphism.

Lemma 7. *The structure $T(H)$ is isomorphic to the completion by cuts of the structure $S(H)$.*

* In the sense of [2], p. 93; from the same source, in the proof, the notations I^* and I^+ will be used.

Proof. Let $\{X_\alpha\}$ be the set of all one-dimensional subspaces from the subspace $H_1 \in T(H)$. Then

$H_1 = \overset{T}{\sup}\{X_\alpha\}$. Next, let $H_1^0 = H_1' = \overset{T}{\sup}\{Y_\beta\}$ be a representation of the subspace H_1^0 as a sum of one-dimensional subspaces Y_β. Then, taking into account Proposition 2 and Lemmas 4 and 5, we obtain

$H_1 = (H_1')' = \overset{T}{\inf}\{Y_\beta'\} = \overset{T}{\inf}\{Y_\beta^0\}$. Thus, the conditions of Lemma 6 are satisfied.

Lemma 8. *Let H be an infinite-dimensional space. Then the structure $T(H)$ is not Dedekind.*

Proof. Let H_1 and H_2 be closed subspaces whose sum H_3 is not closed ((5), Ch. 1). Let $H_4 = H_1 \overset{T}{+} H_2$. Then there exists a vector $a \in H_4 \setminus H_3$. Consider the subspace $H_5 = H_1 \overset{L}{+} \{a\}$. By Proposition 1 we obtain $H_5 \in T(H)$. Moreover, it is clear that

$H_4 = H_1 \overset{T}{+} H_2 = H_5 \overset{T}{+} H_2$. Next, let $H_6 = H_1 \cap H_2$. We shall show that $H_6 = H_5 \cap H_2$. Suppose the contrary; let the vector $b \in (H_5 \cap H_2) \setminus H_6$. Then $b \in H_2 \subseteq H_3$; moreover $b \in H_5$, i.e., $b = h_1 + \alpha a$ ($h_1 \in H_1$, α is a number), and from $b \notin H_6$ it follows that $\alpha \neq 0$. This, however, is impossible, since from $b \in H_3$ and $(-h_1) \in H_1 \subseteq H_3$ it follows that $(-h_1 + b) \in H_3$, but $(-h_1) + b = (-h_1) + h_1 + \alpha a = \alpha a \notin H_3$. Consequently, $H_1 \cap H_2 = H_5 \cap H_2 = H_6$.

From Theorem 1, Corollaries 2 and 4, and Lemmas 7 and 8 it follows:

Theorem 2. *There exist Dedekind structures with orthocomplements whose completion by sections is not a Dedekind structure.*

The author expresses gratitude to Prof. L. A. Skornyakov for a number of useful suggestions.

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Received
1 II 1961

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Note: Figure translations are in progress. See original paper for figures.